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IMSL Fortran Library User's Guide MATH/LIBRARY Special Functions

Mathematical Functions in Fortran

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IMSL MATH/LIBRARY User's Guide Special Functions

Mathematical Functions in Fortran

Visual Numerics

P/N 7681

[w w w . v n i . c o m]

Contents

Introduction

The IMSL Fortran Libraries

The IMSL Libraries consist of two separate, but coordinated Libraries that allow easy user access. These Libraries are organized as follows:

- MATH/LIBRARY general applied mathematics and special functions

  The User's Guide for IMSL MATH/LIBRARY has two parts:

  1. MATH/LIBRARY (Volumes 1 and 2)
  2. MATH/LIBRARY Special Functions

- STAT/LIBRARY statistics

Most of the routines are available in both single and double precision versions. Many routines are also available for complex and complex-double precision arithmetic. The same user interface is found on the many hardware versions that span the range from personal computer to supercomputer. Note that some IMSL routines are not distributed for FORTRAN compiler environments that do not support double precision complex data. The specific names of the IMSL routines that return or accept the type double complex begin with the letter "`Z`" and, occasionally, "`DC`."

Getting Started

IMSL MATH/LIBRARY Special Functions is a collection of FORTRAN subroutines and functions useful in research and statistical analysis. Each routine is designed and documented to be used in research activities as well as by technical specialists.

To use any of these routines, you must write a program in FORTRAN (or possibly some other language) to call the MATH/LIBRARY Special Functions routine. Each routine conforms to established conventions in programming and documentation. We give first priority in development to efficient algorithms, clear documentation, and accurate results. The uniform design of the routines makes it easy to use more than one routine in a given application. Also, you will find that the design consistency enables you to apply your experience with one MATH/LIBRARY Special Functions routine to all other IMSL routines that you use.

Finding the Right Routine

The organization of IMSL MATH/LIBRARY Special Functions closely parallels that of the National Bureau of Standards' *Handbook of Mathematical Functions*, edited by Abramowitz and Stegun (1964). Corresponding to the NBS Handbook, functions are arranged into separate chapters, such as elementary functions, trigonometric and hyperbolic functions, exponential integrals, gamma function and related functions, and Bessel functions. To locate the right routine for a given problem, you may use either the table of contents located in each chapter introduction, or one of the indexes at the end of this manual. GAMS index uses GAMS classification (Boisvert, R.F., S.E. Howe, D.K. Kahaner, and J.L. Springmann 1990, *Guide to Available Mathematical Software*, National Institute of Standards and Technology NISTIR 90-4237). Use the GAMS index to locate which MATH/LIBRARY Special Functions routines pertain to a particular topic or problem.

Organization of the Documentation

This manual contains a concise description of each routine, with at least one demonstrated example of each routine, including sample input and results. You will find all information pertaining to the Special Functions Library in this manual. Moreover, all information pertaining to a particular routine is in one place within a chapter.

Each chapter begins with an introduction followed by a table of contents that lists the routines included in the chapter. Documentation of the routines consists of the following information:

- IMSL Routine's Generic Name
- Purpose: a statement of the purpose of the routine. If the routine is a function rather than a subroutine the purpose statement will reflect this fact.
- Function Return Value: a description of the return value (for functions only).
- Required Arguments: a description of the required arguments in the order of their occurrence. Input arguments usually occur first, followed by input/output arguments, with output arguments described last. Futhermore, the following terms apply to arguments:

  Input Argument must be initialized; it is not changed by the routine.

  Input/Output Argument must be initialized; the routine returns output through this argument; cannot be a constant or an expression.

  Input or Output Select appropriate option to define the argument as either input or output. See individual routines for further instructions.

  Output No initialization is necessary; cannot be a constant or an expression. The routine returns output through this argument.
- Optional Arguments: a description of the optional arguments in the order of their occurrence.
- Fortran 90 Interface: a section that describes the generic and specific interfaces to the routine.
- Fortran 77 Style Interfaces: an optional section, which describes Fortran 77 style interfaces, is supplied for backwards compatibility with previous versions of the Library.

- Example: at least one application of this routine showing input and required dimension and type statements.
- Output: results from the example.
- Comments: details pertaining to code usage.
- Description: a description of the algorithm and references to detailed information. In many cases, other IMSL routines with similar or complementary functions are noted.
- Programming notes: an optional section that contains programming details not covered elsewhere.
- References: periodicals and books with details of algorithm development.
- Additional Examples: an optional section with additional applications of this routine showing input and required dimension and type statements.

Naming Conventions

The names of the routines are mnemonic and unique. Most routines are available in both a single precision and a double precision version, with names of the two versions sharing a common root. The root name is also the generic interface name. The name of the double precision specific version begins with a "D_." The single precision specific version begins with an "S_". For example, the following pairs are precision specific names of routines in the two different precisions: S_GAMDF/D_GAMDF (the root is "GAMDF ," for "Gamma distribution function") and S_POIDF/D_POIDF (the root is "POIDF," for "Poisson distribution function"). The precision specific names of the IMSL routines that return or accept the type complex data begin with the letter "C_" or "Z_" for complex or double complex, respectively. Of course the generic name can be used as an entry point for all precisions supported.

When this convention is not followed the generic and specific interfaces are noted in the documentation. For example, in the case of the BLAS and trigonometric intrinsic functions where standard names are already established, the standard names are used as the precision specific names. There may also be other interfaces supplied to the routine to provide for backwards compatibility with previous versions of the Library. These alternate interfaces are noted in the documentation when they are available.

Except when expressly stated otherwise, the names of the variables in the argument lists follow the FORTRAN default type for integer and floating point. In other words, a variable whose name begins with one of the letters "I" through "N" is of type INTEGER, and otherwise is of type REAL or DOUBLE PRECISION, depending on the precision of the routine.

An assumed-size array with more than one dimension that is used as a FORTRAN argument can have an assumed-size declarator for the last dimension only. In the MATH/LIBRARY Special Functions routines, the information about the first dimension is passed by a variable with the prefix "LD" and with the array name as the root. For example, the argument LDA contains the leading dimension of array *A*. In most cases, information about the dimensions of arrays is obtained from the array through the use of Fortran 90's *size* function. Therefore, arguments carrying this type of information are usually defined as optional arguments.

Where appropriate, the same variable name is used consistently throughout a chapter in the MATH/LIBRARY Special Functions. For example, in the routines for random number generation, `NR` denotes the number of random numbers to be generated, and `R` or `IR` denotes the array that stores the numbers.

When writing programs accessing the MATH/LIBRARY Special Functions , the user should choose FORTRAN names that do not conflict with names of IMSL subroutines, functions, or named common blocks. The careful user can avoid any conflicts with IMSL names if, in choosing names, the following rules are observed:

- Do not choose a name that appears in the Alphabetical Summary of Routines, at the end of the *User's Manual*, nor one of these names preceded by a `D`, `S_`, `D_`, `C_`, or `Z_`.
- Do not choose a name consisting of more than three characters with a numeral in the second or third position.

For further details, see the section on "Reserved Names" in the Reference Material.

Using Library Subprograms

The documentation for the routines uses the generic name and omits the prefix, and hence the entire suite of routines for that subject is documented under the generic name.

Examples that appear in the documentation also use the generic name. To further illustrate this principle, note the `BSJNS` documentation (see Chapter 6, Bessel Functions, of this manual). A description is provided for just one data type. There are four documented routines in this subject area: `S_BSJNS`, `D_BSJNS`, `C_BSJNS`, and `Z_BSJNS`.

These routines constitute single-precision, double-precision, complex, and complex double-precision versions of the code.

The appropriate routine is identified by the Fortran 90 compiler. Use of a module is required with the routines. The naming convention for modules joins the suffix "`_int`" to the generic routine name. Thus, the line "use `BSJNS_INT`" is inserted near the top of any routine that calls the subprogram "`BSJNS`". More inclusive modules are also available. For example, the module named "imsl_libraries" contains the interface modules for all routines in the library.

When dealing with a complex matrix, all references to the *transpose* of a matrix, A^T, are replaced by the *adjoint* matrix

$$\overline{A}^T \equiv A^* = A^H$$

where the overstrike denotes complex conjugation. IMSL Fortran Library linear algebra software uses this convention to conserve the utility of generic documentation for that code subject. References to *orthogonal* matrices are replaced by their complex counterparts, *unitary* matrices. Thus, an $n \times n$ orthogonal matrix Q satisfies the condition $Q^T Q = I_n$. An $n \times n$ unitary matrix V satisfies the analogous condition for complex matrices, $V^* V = I_n$.

Programming Conventions

In general, the IMSL MATH/LIBRARY Special Functions codes are written so that computations are not affected by underflow, provided the system (hardware or software) places a zero value in the register. In this case, system error messages indicating underflow should be ignored.

IMSL codes also are written to avoid overflow. A program that produces system error messages indicating overflow should be examined for programming errors such as incorrect input data, mismatch of argument types, or improper dimensioning.

In many cases, the documentation for a routine points out common pitfalls that can lead to failure of the algorithm.

Library routines detect error conditions, classify them as to severity, and treat them accordingly. This error-handling capability provides automatic protection for the user without requiring the user to make any specific provisions for the treatment of error conditions. See the section on "User Errors" in the Reference Material for further details.

Module Usage

Users are required to incorporate a "use" statement near the top of their program for the IMSL routine being called when writing new code that uses this library. However, legacy code which calls routines in the previous version of the library without the presence of a "use" statement will continue to work as before. The example programs throughout this manual demonstrate the syntax for including use statements in your program. In addition to the examples programs, common cases of when and how to employ a use statement are described below.

- Users writing new programs calling the generic interface to IMSL routines must include a use statement near the top of any routine that calls the IMSL routines. The naming convention for modules joins the suffix "`_int`" to the generic routine name. For example, if a new program is written calling the IMSL routines `LFTRG` and `LFSRG`, then the following use statements should be inserted inserted near the top of the program

  ```
  USE LFTRG_INT
  USE LFSRG_INT
  ```

  In addition to providing interface modules for each routine individually, we also provide a module named "`imsl_libraries`", which contains the generic interfaces for all routines in the library. For programs that call several different IMSL routines using generic interfaces, it can be simpler to insert the line

  ```
  USE IMSL_LIBRARIES
  ```

  rather than list use statements for every IMSL subroutine called.

- Users wishing to update existing programs to call other routines from this library should incorporate a use statement for the new routine being called. (Here, the term "new routine" implies any routine in the library, only "new" to the user's program.) For example, if a call to the generic interface for the routine `LSARG` is added to an existing program, then

  ```
  USE LSARG_INT
  ```

  should be inserted near the top of your program.

- Users wishing to update existing programs to call the new generic versions of the routines must change their calls to the existing routines to match the new calling sequences and use either the routine specific interface modules or the all encompassing "`imsl_libraries`" module.

- Code which employed the "`use numerical_libraries`" statement from the previous version of the library will continue to work properly with this version of the library.

Programming Tips

It is strongly suggested that users force all program variables to be explicitly typed. This is done by including the line "`IMPLICIT NONE`" as close to the first line as possible. Study some of the examples accompanying an IMSL Fortran Library routine early on. These examples are available online as part of the product.

Each subject routine called or otherwise referenced requires the "`use`" statement for an interface block designed for that subject routine. The contents of this interface block are the interfaces to the separate routines available for that subject. Packaged descriptive names for option numbers that modify documented optional data or internal parameters might also be provided in the interface block. Although this seems like an additional complication, many typographical errors are avoided at an early stage in development through the use of these interface blocks. The "`use`" statement is required for each routine called in the user's program.

However, if one is only using the Fortran 77 interfaces supplied for backwards compatibility then the "use" statements are not required.

Optional Subprogram Arguments

IMSL Fortran Library routines have *required* arguments and may have *optional* arguments. All arguments are documented for each routine. For example, consider the routine `GCIN` that evaluates the inverse of a general continuous `CDF`. The required arguments are `P`, `X`, and `F`. The optional arguments are `IOPT` and `M`. Both `IOPT` and `M` take on default values so are not required as input by the user unless the user wishes for these arguments to take on some value other than the default. Often there are other output arguments that are listed as optional because although they may contain information that is closely connected with the computation they are not as compelling as the primary problem. In our example code, `GCIN`, if the user wishes to input the optional argument "`IOPT`" then the use of the keyword "`IOPT=`" in the argument list to assign an input value to `IOPT` would be necessary.

For compatibility with previous versions of the IMSL Libraries, the `NUMERICAL_LIBRARIES` interface module includes backwards compatible positional argument interfaces to all routines which existed in the Fortran 77 version of the Library. Note that it is not necessary to use "use" statements when calling these routines by themselves. Existing programs which called these routines will continue to work in the same manner as before.

Error Handling

The routines in IMSL MATH/LIBRARY Special Functions attempt to detect and report errors and invalid input. Errors are classified and are assigned a code number. By default, errors of moderate or worse severity result in messages being automatically printed by the routine. Moreover, errors of worse severity cause program execution to stop. The severity level as well as the general nature of the error is designated by an "error type" with numbers from 0 to 5. An error type 0 is no error; types 1 through 5 are progressively more severe. In most cases, you need not be concerned with our method of handling errors. For those interested, a complete description of the error-handling system is given in the Reference Material, which also describes how you can change the default actions and access the error code numbers.

Printing Results

None of the routines in IMSL MATH/LIBRARY Special Functions print results (but error messages may be printed). The output is returned in FORTRAN variables, and you can print these yourself.

The IMSL routine `UMACH` (see the Reference Material section of this manual) retrieves the FORTRAN device unit number for printing. Because this routine obtains device unit numbers, it can be used to redirect the input or output. The section on “Machine-Dependent Constants” in the Reference Material contains a description of the routine `UMACH`.

Chapter 1: Elementary Functions

Routines

Usage Notes

The "relative" function `EXPRL` (page 3) is useful for accurately computing $e^x - 1$ near $x = 0$. Computing $e^x - 1$ using `EXP(X)` – 1 near $x = 0$ is subject to large cancellation errors.

Similarly, `ALNREL` (page 5) can be used to accurately compute $\ln(x + 1)$ near $x = 0$. Using the routine `ALOG` to compute $\ln(x + 1)$ near $x = 0$ is subject to large cancellation errors in the computation of 1 + `X`.

CARG

This function evaluates the argument of a complex number.

Function Return Value

CARG — Function value. (Output)
If $z = x + iy$, then $\arctan(y/x)$ is returned except when both x and y are zero. In this case, zero is returned.

Required Arguments

Z — Complex number for which the argument is to be evaluated. (Input)

FORTRAN 90 Interface

Generic: `CARG (Z)`

Specific: The specific interface names are S_CARG and D_CARG.

FORTRAN 77 Interface

Single: CARG (Z)

Double: The double precision function name is ZARG.

Example

In this example, Arg(1 + *i*) is computed and printed.

```
      USE CARG_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE
      COMPLEX    Z
!                                 Compute
      Z     = (1.0, 1.0)
      VALUE = CARG(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CARG(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
CARG( 1.000, 1.000) =  0.785
```

Description

Arg(z) is the angle θ in the polar representation $z = |z|\, e^{i\theta}$, where

$$i = \sqrt{-1}$$

If $z = x + iy$, then $\theta = \tan^{-1}(y/x)$ except when both x and y are zero. In this case, θ is defined to be zero.

CBRT

This funcion evaluates the cube root.

Function Return Value

CBRT — Function value. (Output)

Required Arguments

X — Argument for which the cube root is desired. (Input)

FORTRAN 90 Interface

Generic: `CBRT (X)`

Specific: The specific interface names are `S_CBRT, D_CBRT, C_CBRT, AND Z_CBRT.`

FORTRAN 77 Interface

Single: `CBRT (X)`

Double: The double precision name is `DCBRT`.

Complex: The complex precision name is `CCBRT.`

Double Complex: The Double complex precision name is `ZCBRT.`

Example

In this example, the cube root of 3.45 is computed and printed.

```
      USE CBRT_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 3.45
      VALUE = CBRT(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CBRT(', F6.3, ') = ', F6.3)
      END
```

Output

```
CBRT( 3.450) =  1.511
```

Comments

For complex arguments, the branch cut for the cube root is taken along the negative real axis. The argument of the result, therefore, is greater than $-\pi/3$ and less than or equal to $\pi/3$. The other two roots are obtained by rotating the principal root by $2\pi/3$ and $\pi/3$.

Description

The function `CBRT(X)` evaluates $x^{1/3}$. All arguments are legal. For complex argument, x, the value of $|x|$ must not overflow.

Additional Example

In this example, the cube root of $-3 + 0.0076i$ is computed and printed.

```
      USE UMACH_INT
      USE CBRT_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (-3.0, 0.0076)
      VALUE = CBRT(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (’ CBRT((’, F7.4, ’,’, F7.4, ’)) = (’, &
          F6.3, ’,’, F6.3, ’)’)
      END
```

Output

```
CBRT((-3.0000, 0.0076)) = ( 0.722, 1.248)
```

EXPRL

This function evaluates the exponential function factored from first order, (`EXP`(`X`) – 1.0)/`X`.

Function Return Value

EXPRL — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `EXPRL (X)`

Specific: The specific interface names are `S_EXPRL`, `D_EXPRL`, and `C_EXPRL`.

FORTRAN 77 Interface

Single: `EXPRL (X)`

Double: The double precision function name is `DEXPRL`.

Complex: The complex name is `CEXPRL`.

Example

In this example, `EXPRL`(0.184) is computed and printed.

```
      USE EXPRL_INT
      USE UMACH_INT
```

```
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.184
      VALUE = EXPRL(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' EXPRL(', F6.3, ') = ', F6.3)
      END
```

Output

```
EXPRL( 0.184) =  1.098
```

Description

The function `EXPRL`(`X`) evaluates $(e^x - 1)/x$. It will overflow if e^x overflows. For complex arguments, z, the argument z must not be so close to a multiple of $2\pi i$ that substantial significance is lost due to cancellation. Also, the result must not overflow and $|\Im z|$ must not be so large that the trigonometric functions are inaccurate.

Additional Example

In this example, `EXPRL`(0.0076*i*) is computed and printed.

```
      USE EXPRL_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (0.0, 0.0076)
      VALUE = EXPRL(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' EXPRL((', F7.4, ',', F7.4, ')) = (', &
          F6.3, ',' F6.3, ')')
      END
```

Output

```
EXPRL(( 0.0000, 0.0076)) = ( 1.000, 0.004)
```

LOG10

This function extends FORTRAN's generic log10 function to evauate the principal value of the complex common logarithm.

Function Return Value

LOG10 — Complex function value. (Output)

Required Arguments

Z — Complex argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `LOG10 (Z)`

Specific: The specific interface names are `CLOG10` and `ZLOG10`.

FORTRAN 77 Interface

Complex: `CLOG10 (Z)`

Double complex: The double complex function name is `ZLOG10`.

Example

In this example, the $\log_{10}(0.0076i)$ is computed and printed.

```
      USE LOG10_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                  Compute
      Z     = (0.0, 0.0076)
      VALUE = LOG10(Z)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' LOG10((', F7.4, ',', F7.4, ')) = (', &
              F6.3, ',', F6.3, ')')
      END
```

Output

```
LOG10(( 0.0000, 0.0076)) = (-2.119, 0.682)
```

Description

The function `LOG10(Z)` evaluates $\log_{10}(z)$. The argument must not be zero, and $|z|$ must not overflow.

ALNREL

This function evaluates the natural logarithm of one plus the argument, or, in the case of complex argument, the principal value of the complex natural logarithm of one plus the argument.

Function Return Value

ALNREL — Function value. (Output)

Required Arguments

X — Argument for the function. (Input)

FORTRAN 90 Interface

Generic: ALNREL (X)

Specific: The specific interface names are S_ALNREL, D_ALNREL, and C_ALNREL.

FORTRAN 77 Interface

Single: ALNREL (X)

Double: The double precision name function is DLNREL.

Complex: The comlpex name is CLNREL.

Example

In this example, ln(1.189) = ALNREL(0.189) is computed and printed.

```
      USE ALNREL_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = 0.189
      VALUE = ALNREL(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ALNREL(', F6.3, ') = ', F6.3)
      END
```

Output

```
ALNREL( 0.189) =  0.173
```

Comments

1. Informational error

Type	Code	
3	2	Result of ALNREL(X) is accurate to less than one-half precision because X is too near –1.0.

2. ALNREL evaluates the natural logarithm of (1 + X) accurate in the sense of relative error even when X is very small. This routine (as opposed to the intrinsic ALOG) should be used to maintain relative accuracy whenever X is small and accurately known.

Description

For real arguments, the function ALNREL(X) evaluates $\ln(1 + x)$ for $x > -1$. The argument x must be greater than –1.0 to avoid evaluating the logarithm of zero or a negative number. In addition, x must not be so close to –1.0 that considerable significance is lost in evaluating $1 + x$.

For complex arguments, the function CLNREL(Z) evaluates $\ln(1 + z)$. The argument z must not be so close to –1 that considerable significance is lost in evaluating $1 + z$. If it is, a recoverable error is issued; however, $z = -1$ is a fatal error because $\ln(1 + z)$ is infinite. Finally, $|z|$ must not overflow.

Let $\rho = |z|$, $z = x + iy$ and $r^2 = |1 + z|^2 = (1 + x)^2 + y^2 = 1 + 2x + \rho^2$. Now, if ρ is small, we may evaluate CLNREL(Z) accurately by

$$\begin{aligned}\log(1 + z) &= \log r + i\mathrm{Arg}(z + 1)\\ &= 1/2 \log r^2 + i\mathrm{Arg}(z + 1)\\ &= 1/2\ \mathtt{ALNREL}(2x + \rho^2) + i\mathtt{CARG}(1 + z)\end{aligned}$$

Additional Example

In this example, $\ln(0.0076i)$ = ALNREL($-1 + 0.0076i$) is computed and printed.

```
      USE UMACH_INT
      USE ALNREL_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (-1.0, 0.0076)
      VALUE = ALNREL(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' ALNREL((', F8.4, ',', F8.4, ')) = (', &
          F8.4, ',', F8.4, ')')
      END
```

Output

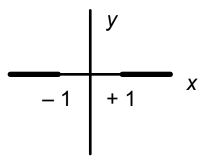
```
 ALNREL((-1.000, .0076)) = (-4.880, 1.571)
```

Chapter 2: Trigonometric and Hyperbolic Functions

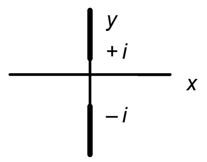
Routines

Usage Notes

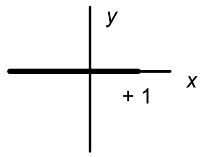
The complex inverse trigonometric hyperbolic functions are single-valued and regular in a slit complex plane. The branch cuts are shown below for $z = x + iy$, i.e., $x = \Re z$ and $y = \Im z$ are the real and imaginary parts of z, respectively.



$\sin^{-1}z$, $\cos^{-1}z$ and $\tanh^{-1}(z)$



$\tan^{-1}z$ and $\sinh^{-1}z$



$\cosh^{-1}z$

Branch Cuts for Inverse Trigonometric and Hyperbolic Functions

TAN

This function extends FORTRAN's generic tan to evaluate the complex tangent.

Function Return Value

TAN — Complex function value. (Output)

Required Arguments

Z — Complex number representing the angle in radians for which the tangent is desired. (Input)

FORTRAN 90 Interface

Generic: `TAN(Z)`

Specific: The specific interface names are `CTAN` and `ZTAN`.

FORTRAN 77 Interface

Complex : `CTAN(Z)`

Double complex: The double complex function name is `ZTAN`.

Example

In this example, $\tan(1 + i)$ is computed and printed.

```
      USE TAN_INT
      USE UMACH_INT
!                                 Declare variables
```

```
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.0, 1.0)
      VALUE = TAN(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' TAN((', F6.3, ',', F6.3, ')) = (', &
          F6.3, ',', F6.3, ')')
      END
```

Output

```
TAN(( 1.000, 1.000)) = ( 0.272, 1.084)
```

Comments

Informational error

Type	Code	
3	2	Result of `CTAN(Z)` is accurate to less than one-half precision because the real part of `Z` is too near $\pi/2$ or $3\pi/2$ when the imaginary part of `Z` is near zero or because the absolute value of the real part is very large and the absolute value of the imaginary part is small.

Description

Let $z = x + iy$. If $|\cos z|^2$ is very small, that is, if x is very close to $\pi/2$ or $3\pi/2$ and if y is small, then $\tan z$ is nearly singular and a fatal error condition is reported. If $|\cos z|^2$ is somewhat larger but still small, then the result will be less accurate than half precision. When $2x$ is so large that $\sin 2x$ cannot be evaluated to any nonzero precision, the following situation results. If $|y| < 3/2$, then `CTAN` cannot be evaluated accurately to better than one significant figure. If $3/2 \le |y| < -1/2 \ln \varepsilon/2$, then `CTAN` can be evaluated by ignoring the real part of the argument; however, the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

COT

This function evaluates the cotangent.

Function Value Return

COT — Function value. (Output)

Required Arguments

X — Angle in radians for which the cotangent is desired. (Input)

FORTRAN 90 Interface

Generic: `COT (X)`

Specific: The specific interface names are COT, DCOT, CCOT, and ZCOT.

FORTRAN 77 Interface

Single: COT (X)

Double: The double precision function name is DCOT.

Complex: The complex name is CCOT.

Double Complex: The double complex name is ZCOT.

Example

In this example, cot(0.3) is computed and printed.

```
      USE COT_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.3
      VALUE = COT(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' COT(', F6.3, ') = ', F6.3)
      END
```

Output

```
COT( 0.300) = 3.233
```

Comments

1. Informational error for Real arguments:

Type	Code	
3	2	Result of COT(X) is accurate to less than one-half precision because ABS(X) is too large, or X is nearly a multiple of π.

Informational error for complex arguments

Type	Code	
3	2	Result of CCOT(Z) is accurate to less than one-half precision because the real part of Z is too near a multiple of π when the imaginary part of Z is zero, or because the absolute value of the real part is very large and the absolute value of the imaginary part is small

2. Referencing COT(X) is NOT the same as computing 1.0/TAN(X) because the error conditions are quite different. For example, when X is near π/2, TAN(X) cannot be

evaluated accurately and an error message must be issued. However, `COT(X)` can be evaluated accurately in the sense of absolute error.

Description

For real x, the magnitude of x must not be so large that most of the computer word contains the integer part of x. Likewise, x must not be too near an integer multiple of π, although x close to the origin causes no accuracy loss. Finally, x must not be so close to the origin that `COT(X)` $\approx 1/x$ overflows.

For complex arguments, let $z = x + iy$. If $|\sin z|^2$ is very small, that is, if x is very close to a multiple of π and if $|y|$ is small, then $\cot z$ is nearly singular and a fatal error condition is reported. If $|\sin z|^2$ is somewhat larger but still small, then the result will be less accurate than half precision. When $|2x|$ is so large that $\sin 2x$ cannot be evaluated accurately to even zero precision, the following situation results. If $|y| < 3/2$, then `CCOT` cannot be evaluated accurately to be better than one significant figure. If $3/2 \le |y| < -1/2 \ln \varepsilon/2$, where ε = `AMACH(4)` is the machine precision, then `CCOT` can be evaluated by ignoring the real part of the argument; however, the answer will be less accurate than half precision. Finally, $|z|$ must not be so small that $\cot z \approx 1/z$ overflows.

Additional Example

In this example, $\cot(1 + i)$ is computed and printed.

```
      USE COT_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.0, 1.0)
      VALUE = COT(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' COT((', F6.3, ',', F6.3, ')) = (', &
          F6.3, ',', F6.3, ')')
      END
```

Output

```
COT(( 1.000, 1.000)) = ( 0.218,-0.868)
```

SINDG

This function evaluates the sine for the argument in degrees.

Function Return Value

SINDG — Function value. (Output)

Required Arguments

X — Argument in degrees for which the sine is desired. (Input)

FORTRAN 90 Interface

Generic: `SINDG (X)`

Specific: The specific interface names are `S_SINDG` and `D_SINDG`.

FORTRAN 77 Interface

Single: `SINDG (X)`

Double: The double precision function name is `DSINDG`.

Example

In this example, sin 45° is computed and printed.

```
      USE SINDG_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = 45.0
      VALUE = SINDG(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SIN(', F6.3, ' deg) = ', F6.3)
      END
```

Output

```
SIN(45.000 deg) = 0.707
```

Description

To avoid unduly inaccurate results, the magnitude of x must not be so large that the integer part fills more than the computer word. Under no circumstances is the magnitude of x allowed to be larger than the largest representable integer because complete loss of accuracy occurs in this case.

COSDG

This function evaluates the cosine for the argument in degrees.

Function Return Value

***COSDG* —** Function value. (Output)

Required Arguments

X — Argument in degrees for which the cosine is desired. (Input)

FORTRAN 90 Interface

Generic: `COSDG (X)`

Specific: The specific interface names are `S_COSDG` and `D_COSDG`.

FORTRAN 77 Interface

Single: `COSDG (X)`

Double: The double precision function name is `DCOSDG`.

Example

In this example, cos 100° computed and printed.

```
      USE COSDG_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 100.0
      VALUE = COSDG(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' COS(', F6.2, ' deg) = ', F6.3)
      END
```

Output

```
COS(100.00 deg) = -0.174
```

Description

To avoid unduly inaccurate results, the magnitude of x must not be so large that the integer part fills more than the computer word. Under no circumstances is the magnitude of x allowed to be larger than the largest representable integer because complete loss of accuracy occurs in this case.

ASIN

This function extends FORTRAN's generic `ASIN` function to evaluate the complex arc sine.

Function Return Value

ASIN — Complex function value in units of radians and the real part in the first or fourth quadrant. (Output)

Required Arguments

ZINP — Complex argument for which the arc sine is desired. (Input)

FORTRAN 90 Interface

Generic: `ASIN(ZINP)`

Specific: The specific interface names are `CASIN` and `ZASIN`.

FORTRAN 77 Interface

Complex: `CASIN(ZINP)`

Double complex: The double complex function name is `ZASIN`.

Example

In this example, $\sin^{-1}(1 - i)$ is computed and printed.

```
      USE ASIN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.0, -1.0)
      VALUE = ASIN(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' ASIN((', F6.3, ',', F6.3, ')) = (', &
          F6.3, ',', F6.3, ')')
      END
```

Output

```
ASIN(( 1.000,-1.000)) = ( 0.666,-1.061)
```

Description

Almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur. Here, b = `AMACH`(2) is the largest floating point number. This error is not detected by `ASIN`.

See Pennisi (1963, page 126) for reference.

ACOS

This function extends FORTRAN's generic `ACOS` function evaluate the complex arc cosine.

Function Return Value

ACOS — Complex function value in units of radians with the real part in the first or second quadrant. (Output)

Required Arguments

Z — Complex argument for which the arc cosine is desired. (Input)

FORTRAN 90 Interface

Generic: `ACOS (Z)`

Specific: The specific interface names are `CACOS` and `ZACOS`.

FORTRAN 77 Interface

Complex: `CACOS (Z)`

Double complex: The double complex function name is `ZACOS`.

Example

In this example, $\cos^{-1}(1 - i)$ is computed and printed.

```
      USE ACOS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.0, -1.0)
      VALUE = ACOS(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' ACOS((', F6.3, ',', F6.3, ')) = (', &
          F6.3, ',', F6.3, ')')
      END
```

Output

```
ACOS(( 1.000,-1.000)) = ( 0.905, 1.061)
```

Description

Almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur. Here, b = `AMACH`(2) is the largest floating point number. This error is not detected by `ACOS`.

ATAN

This function extends FORTRAN's generic function `ATAN` to evaluate the complex arc tangent.

Function Return Value

ATAN — Complex function value in units of radians with the real part in the first or fourth quadrant. (Output)

Required Arguments

Z — Complex argument for which the arc tangent is desired. (Input)

FORTRAN 90 Interface

Generic: `ATAN (Z)`

Specific: The specific interface names are `CATAN` and `ZATAN`.

FORTRAN 77 Interface

Complex: `CATAN (Z)`

Double complex: The double complex function name is `ZATAN`.

Example

In this example, $\tan^{-1}(0.01 - 0.01i)$ is computed and printed.

```
      USE ATAN_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                  Compute
      Z     = (0.01, 0.01)
      VALUE = ATAN(Z)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
```

```
99999 FORMAT (' ATAN((', F6.3, ',', F6.3, ')) = (', &
          F6.3, ',', F6.3, ')')
      END
```

Output

```
ATAN(( 0.010, 0.010)) = ( 0.010, 0.010)
```

Comments

Informational error

Type	Code	
3	2	Result of `ATAN(Z)` is accurate to less than one-half precision because $|Z^2|$ is too close to −1.0.

Description

The argument z must not be exactly ± i, because $\tan^{-1} z$ is undefined there. In addition, z must not be so close to ± i that substantial significance is lost.

ATAN2

This function extends FORTRAN's generic function `ATAN2` to evaluate the complex arc tangent of a ratio.

Function Return Value

ATAN2 — Complex function value in units of radians with the real part between −π and π. (Output)

Required Arguments

CSN — Complex numerator of the ratio for which the arc tangent is desired. (Input)

CCS — Complex denominator of the ratio. (Input)

FORTRAN 90 Interface

Generic: `ATAN2(CSN, CCS)`

Specific: The specific interface names are `CATAN2` and `ZATAN2`.

FORTRAN 77 Interface

Complex: `CATAN2(CSN, CCS)`

Double complex: The double complex function name is `ZATAN2`.

Example

In this example,

$$\tan^{-1}\frac{(1/2)+(i/2)}{2+i}$$

is computed and printed.

```
      USE ATAN2_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, X, Y
!                                 Compute
      X     = (2.0, 1.0)
      Y     = (0.5, 0.5)
      VALUE = ATAN2(Y, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Y, X, VALUE
99999 FORMAT (' ATAN2((', F6.3, ',', F6.3, '), (', F6.3, ',', F6.3,&
          ')) = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
ATAN2(( 0.500, 0.500), ( 2.000, 1.000)) = ( 0.294, 0.092)
```

Comments

The result is returned in the correct quadrant (modulo 2π).

Description

Let z_1 = `CSN` and z_2 = `CCS`. The ratio $z = z_1/z_2$ must not be $\pm\, i$ because $\tan^{-1}(\pm\, i)$ is undefined. Likewise, z_1 and z_2 should not both be zero. Finally, z must not be so close to $\pm i$ that substantial accuracy loss occurs.

SINH

This function extends FORTRAN's generic function SINH to evaluate the complex hyperbolic sine.

Function Return Value

SINH — Complex function value. (Output)

Required Arguments

Z — Complex number representing the angle in radians for which the complex hyperbolic sine is desired. (Input)

FORTRAN 90 Interface

Generic: `SINH(Z)`

Specific: The specific interface names are `CSINH` and `ZSINH`.

FORTRAN 77 Interface

Complex: `CSINH(Z)`

Double complex: The double complex function name is `ZSINH`.

Example

In this example, sinh(5 – *i*) is computed and printed.

```
      USE SINH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (5.0, -1.0)
      VALUE = SINH(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' SINH((', F6.3, ',', F6.3, ')) = (',&
            F7.3, ',', F7.3, ')')
      END
```

Output

```
SINH(( 5.000,-1.000)) = ( 40.092,-62.446)
```

Description

The argument *z* must satisfy

$$\left|\Im z\right| \le 1/\sqrt{\varepsilon}$$

where ε = `AMACH`(4) is the machine precision and $\Im z$ is the imaginary part of *z*.

COSH

The function extends FORTRAN's generic function `COSH` to evaluate the complex hyperbolic cosine.

Function Return Value

COSH — Complex function value. (Output)

Required Arguments

Z — Complex number representing the angle in radians for which the hyperbolic cosine is desired. (Input)

FORTRAN 90 Interface

Generic: `COSH (Z)`

Specific: The specific interface names are `CCOSH` and `ZCOSH`.

FORTRAN 77 Interface

Complex: `CCOSH (Z)`

Double complex: The double complex function name is `ZCOSH`.

Example

In this example, cosh(−2 + 2*i*) is computed and printed.

```
      USE COSH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (-2.0, 2.0)
      VALUE = COSH(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' COSH((', F6.3, ',', F6.3, ')) = (',&
           F6.3, ',', F6.3, ')')
      END
```

Output

```
COSH((-2.000, 2.000)) = (-1.566,-3.298)
```

Description

Let ε = AMACH(4) be the machine precision. If $|\Im z|$ is larger than

$$1/\sqrt{\varepsilon}$$

then the result will be less than half precision, and a recoverable error condition is reported. If $|\Im z|$ is larger than 1/ε, the result has no precision and a fatal error is reported. Finally, if $|\Re z|$ is too large, the result overflows and a fatal error results. Here, $\Re z$ and $\Im z$ represent the real and imaginary parts of *z*, respectively.

TANH

This function extends FORTRAN's generic function TANH to evaluate the complex hyperbolic tangent.

Function Return Value

TANH — Complex function value. (Output)

Required Arguments

Z — Complex number representing the angle in radians for which the hyperbolic tangent is desired. (Input)

FORTRAN 90 Interface

Generic: TANH (Z)

Specific: The specific interface names are CTANH and ZTANH.

FORTRAN 77 Interface

Complex: CTANH (Z)

Double complex: The double complex function name is ZTANH.

Example

In this example, tanh(1 + *i*) is computed and printed.

```
      USE TANH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.0, 1.0)
      VALUE = TANH(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' TANH((', F6.3, ',', F6.3, ')) = (',&
            F6.3, ',', F6.3, ')')
      END
```

Output

```
TANH(( 1.000, 1.000)) = ( 1.084, 0.272)
```

Description

Let $z = x + iy$. If $|\cosh z|^2$ is very small, that is, if $y \bmod 2\pi$ is very close to $\pi/2$ or $3\pi/2$ and if x is small, then $\tanh z$ is nearly singular; a fatal error condition is reported. If $|\cosh z|^2$ is somewhat larger but still small, then the result will be less accurate than half precision. When $2y$ ($z = x + iy$) is so large that $\sin 2y$ cannot be evaluated accurately to even zero precision, the following situation results. If $|x| < 3/2$, then `TANH` cannot be evaluated accurately to better than one significant figure. If $3/2 \le |y| < -1/2 \ln (\varepsilon/2)$, then `TANH` can be evaluated by ignoring the imaginary part of the argument; however, the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

ASINH

This function evaluates the arc hyperbolic sine.

Function Return Value

ASINH — Function value. (Output)

Required Arguments

X — Argument for which the arc hyperbolic sine is desired. (Input)

FORTRAN 90 Interface

Generic: `ASINH (X)`

Specific: The specific interface names are `ASINH`, `DASINH`, `CASINH`, and `ZASINH`.

FORTRAN 77 Interface

Single: `ASINH (X)`

Double: The double precision function name is `DASINH`.

Complex: The complex name is `CASINH`.

Double Complex: The double complex name is `ZASINH`

Example

In this example, $\sinh^{-1}(2.0)$ is computed and printed.

```
      USE ASINH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
```

```
      X     = 2.0
      VALUE = ASINH(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ASINH(', F6.3, ') = ', F6.3)
      END
```

Output

```
ASINH( 2.000) = 1.444
```

Description

The function ASINH(X) computes the inverse hyperbolic sine of x, $\sinh^{-1}x$.

For complex arguments, almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur, where b = AMACH(2) is the largest floating point number. This error is not detected by ASINH.

Additional Example

In this example, $\sinh^{-1}(-1 + i)$ is computed and printed.

```
      USE ASINH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (-1.0, 1.0)
      VALUE = ASINH(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' ASINH((', F6.3, ',', F6.3, ')) = (', &
              F6.3, ',', F6.3, ')')
      END
```

Output

```
ASINH((-1.000, 1.000)) = (-1.061, 0.666)
```

ACOSH

This function evaluates the arc hyperbolic cosine.

Function Return Value

ACOSH — Function value. (Output)

Required Arguments

X — Argument for which the arc hyperbolic cosine is desired. (Input)

FORTRAN 90 Interface

Generic: `ACOSH (X)`

Specific: The specific interface names are `ACOSH, DACOSH, CACOSH, and ZACOSH.`

FORTRAN 77 Interface

Single: `ACOSH (X)`

Double: The double precision function name is `DACOSH`.

Complex: The complex name is `CACOSH`.

Double Complex: The double complex name is `ZACOSH`.

Example

In this example, $\cosh^{-1}(1.4)$ is computed and printed.

```
      USE ACOSH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.4
      VALUE = ACOSH(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ACOSH(', F6.3, ') = ', F6.3)
      END
```

Output

```
ACOSH( 1.400) = 0.867
```

Comments

The result of `ACOSH`(`X`) is returned on the positive branch. Recall that, like `SQRT`(`X`), `ACOSH`(`X`) has multiple values.

Description

The function `ACOSH`(`X`) computes the inverse hyperbolic cosine of x, $\cosh^{-1}x$.

For complex arguments, almost all arguments are legal. Only when $|z| > b/2$ can an overflow occur, where b = `AMACH`(2) is the largest floating point number. This error is not detected by `ACOSH`.

Additional Example

In this example, $\cosh^{-1}(1 - i)$ is computed and printed.

```
      USE ACOSH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.0, -1.0)
      VALUE = ACOSH(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' ACOSH((', F6.3, ',', F6.3, ')) = (', &
              F6.3, ',', F6.3, ')')
      END
```

Output

```
ACOSH(( 1.000,-1.000)) = (-1.061, 0.905)
```

ATANH

This function evaluates the arc hyperbolic tangent.

Function Return Value

ATANH — Function value. (Output)

Required Arguments

X — Argument for which the arc hyperbolic tangent is desired. (Input)

FORTRAN 90 Interface

Generic: `ATANH (X)`

Specific: The specific interface names are `ATANH`, `DATANH`, `CATANH`, and `ZATANH`

FORTRAN 77 Interface

Single: `ATANH (X)`

Double: The double precision function name is `DATANH`.

Complex: The complex name is `CATANH`.

Double Complex: The double complex name is `ZATANH`.

Example

In this example, $\tanh^{-1}(-1/4)$ is computed and printed.

```
      USE ATANH_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = -0.25
      VALUE = ATANH(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ATANH(', F6.3, ') = ', F6.3)
      END
```

Output

```
ATANH(-0.250) = -0.255
```

Comments

Informational error

Type	Code	
3	2	Result of ATANH(X) is accurate to less than one-half precision because the absolute value of the argument is too close to 1.0.

Description

ATANH(X) computes the inverse hyperbolic tangent of x, $\tanh^{-1}x$. The argument x must satisfy

$$|x| < 1 - \sqrt{\varepsilon}$$

where ε = AMACH(4) is the machine precision. Note that $|x|$ must not be so close to one that the result is less accurate than half precision.

Additional Example

In this example, $\tanh^{-1}(1/2 + i/2)$ is computed and printed.

```
      USE ATANH_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                  Compute
      Z     = (0.5, 0.5)
      VALUE = ATANH(Z)
```

```
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (’ ATANH((’, F6.3, ’,’, F6.3, ’)) = (’, &
              F6.3, ’,’, F6.3, ’)’)
      END
```

Output

```
 ATANH(( 0.500, 0.500)) = ( 0.402, 0.554)
```

Chapter 3: Exponential Integrals and Related Functions

Routines

Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964).

The following is a plot of the exponential integral functions that can be computed by the routines described in this chapter.

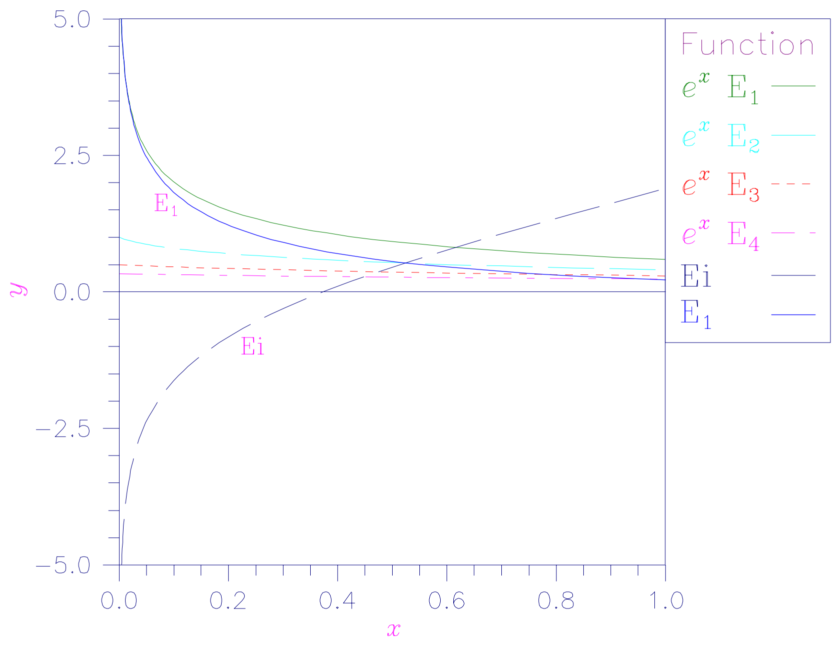


Figure 3-1 Plot of e^xE(x), $E_1(x)$ and Ei(x)

EI

This function evaluates the exponential integral for arguments greater than zero and the Cauchy principal value for arguments less than zero.

Function Return Value

EI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `EI (X)`

Specific: The specific interface names are `S_EI` and `D_EI`.

FORTRAN 77 Interface

Single: `EI (X)`

Double: The double precision function name is `DEI`.

Example

In this example, Ei(1.15) is computed and printed.

```
      USE EI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.15
      VALUE = EI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' EI(', F6.3, ') = ', F6.3)
      END
```

Output

```
EI( 1.150) =  2.304
```

Comments

If principal values are used everywhere, then for all `X`, `EI(X)` = −`E1`(−`X`) and `E1(X)` = −`EI`(−`X`)

Description

The exponential integral, Ei(x), is defined to be

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} e^{-t}/t \; dt \quad \text{for } x \neq 0$$

The argument x must be large enough to insure that the asymptotic formula e^x/x does not underflow, and x must not be so large that e^x overflows.

E1

This function evaluates the exponential integral for arguments greater than zero and the Cauchy principal value of the integral for arguments less than zero.

Function Return Value

E1 — Function value. (Output)

Required Arguments

X — Argument for which the integral is to be evaluated. (Input)

FORTRAN 90 Interface

Generic: `E1 (X)`

Specific: The specific interface names are `S_E1` and `D_E1`.

FORTRAN 77 Interface

Single: `E1 (X)`

Double: The double precision function name is `DE1`.

Example

In this example, $E_1(1.3)$ is computed and printed.

```
      USE E1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.3
      VALUE = E1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' E1(', F6.3, ') = ', F6.3)
      END
```

Output

```
E1( 1.300) =  0.135
```

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Description

The alternate definition of the exponential integral, $E_1(x)$, is

$$E_1(x) = \int_x^\infty e^{-t}/t \, dt \quad \text{for } x \neq 0$$

The path of integration must exclude the origin and not cross the negative real axis.

The argument x must be large enough that e^{-x} does not overflow, and x must be small enough to insure that e^{-x}/x does not underflow.

ENE

Evaluates the exponential integral of integer order for arguments greater than zero scaled by `EXP(X)`.

Required Arguments

X — Argument for which the integral is to be evaluated. (Input)
It must be greater than zero.

N — Integer specifying the maximum order for which the exponential integral is to be calculated. (Input)

F — Vector of length `N` containing the computed exponential integrals scaled by `EXP(X)`. (Output)

FORTRAN 90 Interface

Generic: `CALL ENE (X, N, F)`

Specific: The specific interface names are `S_ENE` and `D_ENE`.

FORTRAN 77 Interface

Single: `CALL ENE (X, N, F)`

Double: The double precision function name is `DENE`.

Example

In this example, $E_n(10)$ for $n = 1, ..., n$ is computed and printed.

```
      USE ENE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
!
      INTEGER    K, NOUT
      REAL       F(N), X
!                                 Compute
      X = 10.0
      CALL ENE (X, N, F)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K, X, F(K)
   10 CONTINUE
99999 FORMAT (' E sub ', I2, ' (', F6.3, ') = ', F6.3)
      END
```

Output

```
E sub  1 (10.000) =  0.092
E sub  2 (10.000) =  0.084
E sub  3 (10.000) =  0.078
E sub  4 (10.000) =  0.073
E sub  5 (10.000) =  0.068
E sub  6 (10.000) =  0.064
E sub  7 (10.000) =  0.060
E sub  8 (10.000) =  0.057
E sub  9 (10.000) =  0.054
E sub 10 (10.000) =  0.051
```

Description

The scaled exponential integral of order n, $E_n(x)$, is defined to be

$$E_n(x) = e^x \int_1^\infty e^{-xt} t^{-n} \, dt \qquad \text{for } x > 0$$

The argument x must satisfy $x > 0$. The integer n must also be greater than zero. This code is based on a code due to Gautschi (1974).

ALI

This function evaluates the logarithmic integral.

Function Return Value

ALI — Function value. (Output)

Required Arguments

X — Argument for which the logarithmic integral is desired. (Input)
It must be greater than zero and not equal to one.

FORTRAN 90 Interface

Generic: `ALI (X)`

Specific: The specific interface names are `S_ALI` and `D_ALI`.

FORTRAN 77 Interface

Single: `ALI (X)`

Double: The double precision function name is `DALI`.

Example

In this example, li(2.3) is computed and printed.

```
      USE ALI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 2.3
      VALUE = ALI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ALI(', F6.3, ') = ', F6.3)
      END
```

Output

```
ALI( 2.300) =  1.439
```

Comments

Informational error

Type	Code	
3	2	Result of `ALI(X)` is accurate to less than one-half precision because `X` is too close to 1.0.

Description

The logarithmic integral, li(x), is defined to be

$$\mathrm{li}(x) = -\int_0^x \frac{dt}{\ln t} \qquad \text{for } x > 0 \text{ and } x \neq 1$$

The argument x must be greater than zero and not equal to one. To avoid an undue loss of accuracy, x must be different from one at least by the square root of the machine precision.

The function li(x) approximates the function $\pi(x)$, the number of primes less than or equal to x. Assuming the Riemann hypothesis (all non-real zeros of $\zeta(z)$ are on the line $\Re z = 1/2$), then

$$\mathrm{li}(x) - \pi(x) = O(\sqrt{x} \ln x)$$

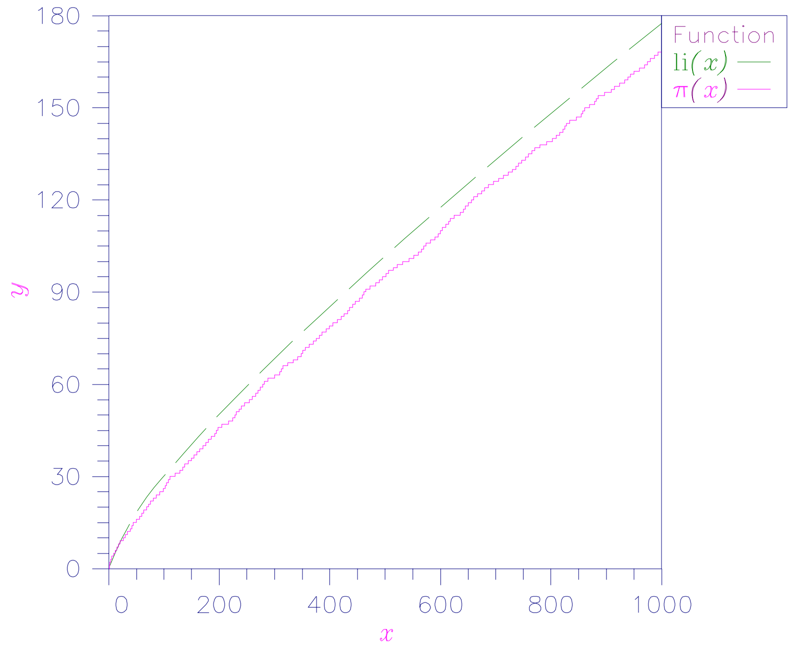


Figure 3-2 Plot of li(x) and π(x)

SI

This function evaluates the sine integral.

Function Return Value

SI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `SI (X)`

Specific: The specific interface names are `S_SI` and `D_SI`.

FORTRAN 77 Interface

Single: `SI` (X)

Double: The double precision function name is `DSI`.

Example

In this example, Si(1.25) is computed and printed.

```
      USE SI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.25
      VALUE = SI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SI(', F6.3, ') = ', F6.3)
      END
```

Output

```
SI( 1.250) =  1.146
```

Description

The sine integral, Si(x), is defined to be

$$\mathrm{Si}(x)=\int_0^x \frac{\sin t}{t}\,dt$$

If

$$|x| > 1/\sqrt{\varepsilon}$$

the answer is less accurate than half precision, while for $|x| > 1/\varepsilon$, the answer has no precision. Here, ε = `AMACH`(4) is the machine precision.

CI

This function evaluates the cosine integral.

Function Return Value

CI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be greater than zero.

FORTRAN 90 Interface

Generic: `CI (X)`

Specific: The specific interface names are `S_CI` and `D_CI`.

FORTRAN 77 Interface

Single: `CI (X)`

Double: The double precision function name is `DCI`.

Example

In this example, Ci(1.5) is computed and printed.

```
      USE CI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.5
      VALUE = CI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CI(', F6.3, ') = ', F6.3)
      END
```

Output

```
CI( 1.500) =  0.470
```

Description

The cosine integral, Ci(x), is defined to be

$$\mathrm{Ci}(x) = \gamma + \ln x + \int_0^x \frac{1-\cos t}{t}\,dt$$

where $\gamma \approx 0.57721566$ is Euler's constant.

The argument x must be larger than zero. If

$$x > 1/\sqrt{\varepsilon}$$

then the result will be less accurate than half precision. If $x > 1/\varepsilon$, the result will have no precision. Here, ε = `AMACH`(4) is the machine precision.

CIN

This function evaluates a function closely related to the cosine integral.

Function Return Value

CIN — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `CIN (X)`

Specific: The specific interface names are `S_CIN` and `D_CIN`.

FORTRAN 77 Interface

Single: `CIN (X)`

Double: The double precision function name is `DCIN`.

Example

In this example, Cin(2π) is computed and printed.

```
      USE CIN_INT
      USE UMACH_INT
      USE CONST_INT
!                                 Declare variables
!
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = CONST('pi')
      X     = 2.0* X
      VALUE = CIN(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CIN(', F6.3, ') = ', F6.3)
      END
```

Output

```
CIN( 6.283) =  2.438
```

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too small.

Description

The alternate definition of the cosine integral, Cin(x), is

$$\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\,dt$$

For

$$0 < |x| < \sqrt{s}$$

where s = `AMACH`(1) is the smallest representable positive number, the result underflows. For

$$|x| > 1/\sqrt{\varepsilon}$$

the answer is less accurate than half precision, while for $|x| > 1/\varepsilon$, the answer has no precision. Here, ε = `AMACH`(4) is the machine precision.

SHI

This function evaluates the hyperbolic sine integral.

Function Return Value

SHI — function value. (Output)
`SHI` equals

$$\int_0^x \sinh(t)/t\,dt$$

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `SHI (X)`

Specific: The specific interface names are `S_SHI` and `D_SHI`.

FORTRAN 77 Interface

Single: `SHI (X)`

Double: The double precision function name is `DSHI`.

Example

In this example, Shi(3.5) is computed and printed.

```
      USE SHI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 3.5
      VALUE = SHI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SHI(', F6.3, ') = ', F6.3)
      END
```

Output

```
SHI( 3.500) =  6.966
```

Description

The hyperbolic sine integral, Shi(x), is defined to be

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t}\,dt$$

The argument x must be large enough that e^{-x}/x does not underflow, and x must be small enough that e^x does not overflow.

CHI

This function evaluates the hyperbolic cosine integral.

Function Return Value

CHI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `CHI (X)`

Specific: The specific interface names are `S_CHI` and `D_CHI`.

FORTRAN 77 Interface

Single: `CHI (X)`

Double: The double precision function name is DCHI.

Example

In this example, Chi(2.5) is computed and printed.

```
      USE CHI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 2.5
      VALUE = CHI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CHI(', F6.3, ') = ', F6.3)
      END
```

Output

```
CHI( 2.500) =  3.524
```

Comments

When X is negative, the principal value is used.

Description

The hyperbolic cosine integral, Chi(x), is defined to be

$$\mathrm{Chi}(x) = \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t}\,dt \qquad \text{for } x > 0$$

where $\gamma \approx 0.57721566$ is Euler's constant.

The argument x must be large enough that e^{-x}/x does not underflow, and x must be small enough that e^x does not overflow.

CINH

This function evaluates a function closely related to the hyperbolic cosine integral.

Function Return Value

CINH — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `CINH (X)`

Specific: The specific interface names are `S_CINH` and `D_CINH`.

FORTRAN 77 Interface

Single: `CINH (X)`

Double: The double precision function name is `DCINH`.

Example

In this example, Cinh(2.5) is computed and printed.

```
      USE CINH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 2.5
      VALUE = CINH(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' CINH(', F6.3, ') = ', F6.3)
      END
```

Output

```
CINH( 2.500) =  2.031
```

Comments

Informational error

Type	Code	
2	1	The function underflows because X is too small.

Description

The alternate definition of the hyperbolic cosine integral, Cinh(x), is

$$\text{Cinh}(x) = \int_0^x \frac{\cosh t - 1}{t} dt$$

For

$$0 < |x| < 2\sqrt{s}$$

where s = `AMACH`(1) is the smallest representable positive number, the result underflows. The argument x must be large enough that e^{-x}/x does not underflow, and x must be small enough that e^x does not overflow.

Chapter 4: Gamma Function and Related Functions

Routines

Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964).

The following is a table of the functions defined in this chapter:

FAC	$n! = \Gamma(n + 1)$
BINOM	$n!/m!(n - m)!$, $0 \le m \le n$
GAMMA	$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, $x \ne 0, -1, -2, \ldots$
GAMR	$1/\Gamma(x)$
ALNGAM	$\ln \lvert\Gamma(x)\rvert$, $x \ne 0, -1, -2, \ldots$
ALGAMS	$\ln \lvert\Gamma(x)\rvert$ and sign $\Gamma(x)$, $x \ne 0, -1, -2, \ldots$
GAMI	$\gamma(a, x) = \int_0^x t^{a-1}e^{-t}\,dt, a > 0$, $x \ge 0$
GAMIC	$\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}\,dt$, $x > 0$
GAMIT	$\gamma^*(a, x) = (x^{-a}/\Gamma(a))\gamma(a, x)$, $x \ge 0$
PSI	$\psi(x) = \Gamma'(x)/\Gamma(x)$, $x \ne 0, -1, -2, \ldots$
POCH	$(a)_x = \Gamma(a + x)/\Gamma(a)$, if $a + x = 0, -1, -2, \ldots$ then a must $= 0, -1, -2, \ldots$
POCH1	$((a)_x - 1)/x$, if $a + x = 0, -1, -2, \ldots$ then a must $= 0, -1, -2, \ldots$
BETA	$\beta(x_1, x_2) = \Gamma(x_1)\Gamma(x_2)/\Gamma(x_1 + x_2)$, $x_1 > 0$ and $x_2 > 0$
CBETA	$\beta(z_1, z_2) = \Gamma(z_1)\Gamma(z_2)/\Gamma(z_1 + z_2)$, $z_1 > 0$ and $z_2 > 0$
ALBETA	$\ln \beta(a, b)$, $a > 0$, $b > 0$
BETAI	$I_x(a, b) = \beta_x(a, b)/\beta(a, b)$, $0 \le x \le 1$, $a > 0$, $b > 0$

FAC

This function evaluates the factorial of the argument.

Function Return Value

FAC — Function value. (Output)
See Comment 1.

Required Arguments

N — Argument for which the factorial is desired. (Input)

FORTRAN 90 Interface

Generic: `FAC (N)`

Specific: The specific interface names are `S_FAC` and `D_FAC`.

FORTRAN 77 Interface

Single: `FAC (N)`

Double: The double precision function name is `DFAC`.

Example

In this example, 6! is computed and printed.

```
      USE FAC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N, NOUT
      REAL       VALUE
!                                 Compute
      N     = 6
      VALUE = FAC(N)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) N, VALUE
99999 FORMAT (' FAC(', I1, ') = ', F6.2)
      END
```

Output

```
FAC(6) = 720.00
```

Comments

1. If the generic version of this function is used, the immediate result must be stored in a variable before use in an expression. For example:

   ```
   X = FAC(6)
   Y = SQRT(X)
   ```

   must be used rather than

   ```
   Y = SQRT(FAC(6)).
   ```

   If this is too much of a restriction on the programmer, then the specific name can be used without this restriction.

To evaluate the factorial for nonintegral values of the argument, the gamma function should be used. For large values of the argument, the log gamma function should be used.

Description

The factorial is computed using the relation $n! = \Gamma(n + 1)$. The function $\Gamma(x)$ is defined in `GAMMA` on page 51. The argument n must be greater than or equal to zero, and it must not be so large that $n!$ overflows. Approximately, $n!$ overflows when $n^n e^{-n}$ overflows.

BINOM

This function evaluates the binomial coefficient.

Function Return Value

BINOM — Function value. (Output)
See Comment 1.

Required Arguments

N — First parameter of the binomial coefficient. (Input)
`N` must be nonnegative.

M — Second parameter of the binomial coefficient. (Input)
`M` must be nonnegative and less than or equal to `N`.

FORTRAN 90 Interface

Generic: `BINOM (N, M)`

Specific: The specific interface names are `S_BINOM` and `D_BINOM`.

FORTRAN 77 Interface

Single: `BINOM (N, M)`

Double: The double precision function name is `DBINOM`.

Example

In this example, $\binom{9}{5}$ is computed and printed.

```
      USE BINOM_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    M, N, NOUT
      REAL       VALUE
!                                 Compute
      N     = 9
      M     = 5
      VALUE = BINOM(N, M)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) N, M, VALUE
99999 FORMAT (' BINOM(', I1, ',', I1, ') = ', F6.2)
      END
```

Output

```
BINOM(9,5) = 126.00
```

Comments

1. If the generic version of this function is used, the immediate result must be stored in a variable before use in an expression. For example:

   ```
   X = BINOM(9, 5)
   Y = SQRT(X)
   ```

   must be used rather than

   ```
   Y = SQRT(BINOM(9, 5)).
   ```

   If this is too much of a restriction on the programmer, then the specific name can be used without this restriction.

2. To evaluate binomial coefficients for nonintegral values of the arguments, the complete beta function or log beta function should be used.

Description

The binomial function is defined to be

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

with $n \geq m \geq 0$. Also, n must not be so large that the function overflows.

GAMMA

This function evaluates the complete gamma function.

Function Return Value

GAMMA — Function value. (Output)

Required Arguments

X — Argument for which the complete gamma function is desired. (Input)

FORTRAN 90 Interface

Generic: `GAMMA (X)`

Specific: The specific interface names are `S_GAMMA`, `D_GAMMA`, and `C_GAMMA`.

FORTRAN 77 Interface

Single: `GAMMA (X)`

Double: The double precision function name is `DGAMMA`.

Complex: The complex name is `CGAMMA`.

Example

In this example, Γ(5.0) is computed and printed.

```
      USE GAMMA_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 5.0
      VALUE = GAMMA(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' GAMMA(', F6.3, ') = ', F6.3)
      END
```

Output

```
GAMMA( 5.000) = 24.000
```

Comments

Informational errors

Type	Code	
2	1	The function underflows because `X` is too small.
3	2	Result is accurate to less than one-half precision because `X` is too near a negative integer.

Description

The gamma function, Γ(*z*), is defined to be

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad \text{for } \Re z > 0$$

For $\Re(z) < 0$, the above definition is extended by analytic continuation.

z must not be so close to a negative integer that the result is less accurate than half precision. If $\Re(z)$ is too small, then the result will underflow. Users who need such values should use the log gamma function `ALNGAM`, page 55. When $\Im(z) \approx 0$, $\Re(z)$ should be greater than $x_{\min}$ so that the result does not underflow, and $\Re(z)$ should be less than $x_{\max}$ so that the result does not overflow. $x_{\min}$ and $x_{\max}$ are available by

```
CALL R9GAML (XMIN, XMAX)
```

Note that z must not be too far from the real axis because the result will underflow.

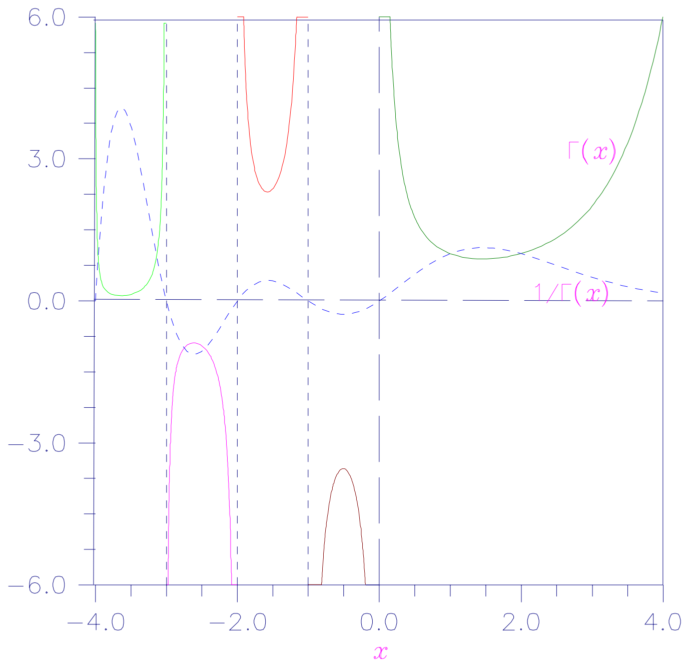


Figure 4-1 Plot of $\Gamma(x)$ and $1/\Gamma(x)$

Additional Example

In this example, $\Gamma(1.4 + 3i)$ is computed and printed.

```
      USE GAMMA_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.4, 3.0)
      VALUE = GAMMA(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' GAMMA(', F6.3, ',', F6.3, ') = (', &
           F6.3, ',', F6.3, ')')
      END
```

Output

```
GAMMA( 1.400, 3.000) = (-0.001, 0.061)
```

GAMR

This function evaluates the reciprocal gamma function.

Function Return Value

GAMR — Function value. (Output)

Required Arguments

X — Argument for which the reciprocal gamma function is desired. (Input)

FORTRAN 90 Interface

Generic: `GAMR (X)`

Specific: The specific interface names are `S_GAMR`, `D_GAMR`, and `C_GAMR`

FORTRAN 77 Interface

Single: `GAMR (X)`

Double: The double precision function name is `DGAMR`.

Complex: The complex name is `CGAMR`.

Example

In this example, 1/Γ(1.85) is computed and printed.

```
      USE GAMR_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = 1.85
      VALUE = GAMR(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' GAMR(', F6.3, ') = ', F6.3)
      END
```

Output

```
GAMR( 1.850) = 1.058
```

Comments

This function is well behaved near zero and negative integers.

Description

The function `GAMR` computes $1/\Gamma(z)$. See `GAMMA` (page 51) for the definition of $\Gamma(z)$.

For $\Im(z) \approx 0$, z must be larger than $x_{\min}$ so that $1/\Gamma(z)$ does not underflow, and x must be smaller than $x_{\max}$ so that $1/\Gamma(z)$ does not overflow. Symmetric overflow and underflow limits $x_{\min}$ and $x_{\max}$ are obtainable from

```
CALL R9GAML (XMIN, XMAX)
```

Note that z must not be too far from the real axis because the result will overflow there.

Additional Example

In this example, $\ln \Gamma(1.4 + 3i)$ is computed and printed.

```
      USE GAMR_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.4, 3.0)
      VALUE = GAMR(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' GAMR(', F6.3, ',', F6.3, ') = (', F7.3, ',', F7.3, ')')
      END
```

Output

```
GAMR( 1.400, 3.000) = ( -0.303,-16.367)
```

ALNGAM

The function evaluates the logarithm of the absolute value of the gamma function.

Function Return Value

ALNGAM — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `ALNGAM (X)`

Specific: The specific interface names are `S_ALNGAM`, `D_ALNGAM`, and `C_ALNGAM`.

FORTRAN 77 Interface

Single:	`ALNGAM (X)`
Double:	The double precision function name is `DLNGAM`.
Complex:	The complex name is `CLNGAM`.

Example

In this example, ln |Γ(1.85)| is computed and printed.

```
      USE ALNGAM_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = 1.85
      VALUE = ALNGAM(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ALNGAM(', F6.3, ') = ', F6.3)
      END
```

Output

```
ALNGAM( 1.850) = -0.056
```

Comments

Informational error

Type	Code	
3	2	Result of `ALNGAM(X)` is accurate to less than one-half precision because `X` is too near a negative integer.

Description

The function `ALNGAM` computes ln |Γ(x)|. See `GAMMA` (page 51) for the definition of Γ(x).

The gamma function is not defined for integers less than or equal to zero. Also, |x| must not be so large that the result overflows. Neither should x be so close to a negative integer that the accuracy is worse than half precision.

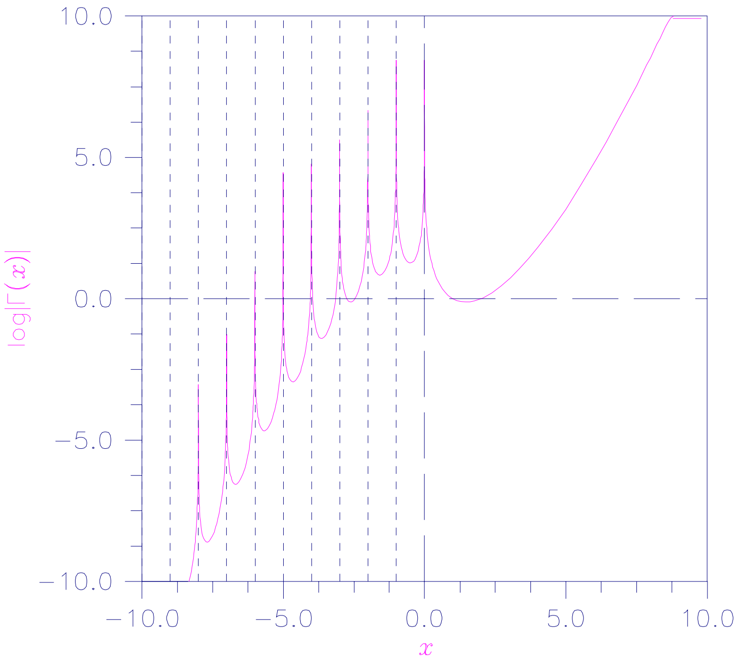


Figure 4-2 Plot of log|Γ(*x*)|

Additional Example

In this example, ln Γ(1.4 + 3*i*) is computed and printed.

```
      USE ALNGAM_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.4, 3.0)
      VALUE = ALNGAM(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' ALNGAM(', F6.3, ',', F6.3, ') = (',&
           F6.3, ',', F6.3, ')')
      END
```

Output

```
ALNGAM( 1.400, 3.000) = (-2.795, 1.589)
```

ALGAMS

Returns the logarithm of the absolute value of the gamma function and the sign of gamma.

Required Arguments

X — Argument for which the logarithm of the absolute value of the gamma function is desired. (Input)

ALGM — Result of the calculation. (Output)

S — Sign of gamma(X). (Output)
If gamma(X) is greater than or equal to zero, S = 1.0. If gamma(X) is less than zero, S = −1.0.

FORTRAN 90 Interface

Generic: `CALL ALGAMS (X, ALGM, S)`

Specific: The specific interface names are `S_ALGAMS` and `D_ALGAMS`.

FORTRAN 77 Interface

Single: `CALL ALGAMS (X, ALGM, S)`

Double: The double precision function name is `DLGAMS`.

Example

In this example, ln $|\Gamma(1.85)|$ and the sign of $\Gamma(1.85)$ are computed and printed.

```
      USE ALGAMS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, S, X
!                                 Compute
      X = 1.85
      CALL ALGAMS(X, VALUE, S)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99998) X, VALUE
99998 FORMAT (' Log Abs(Gamma(', F6.3, ')) = ', F6.3)
      WRITE (NOUT,99999) X, S
99999 FORMAT (' Sign(Gamma(', F6.3, ')) = ', F6.2)
      END
```

Output

```
Log Abs(Gamma( 1.850)) = -0.056
Sign(Gamma( 1.850)) = 1.00
```

Comments

Informational error

Type	Code	
3	2	Result of `ALGAMS` is accurate to less than one-half precision because `X` is too near a negative integer.

Description

The function `ALGAMS` computes ln $|\Gamma(x)|$ and the sign of $\Gamma(x)$. See `GAMMA` (page 51) for the definition of $\Gamma(x)$.

The result overflows if $|x|$ is too large. The accuracy is worse than half precision if x is too close to a negative integer.

GAMI

This funciton evaluates the incomplete gamma function.

Function Return Value

GAMI — Function value. (Output)

Required Arguments

A — The integrand exponent parameter. (Input)
It must be positive.

X — The upper limit of the integral definition of `GAMI`. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `GAMI (A, X)`

Specific: The specific interface names are `S_GAMI` and `D_GAMI`.

FORTRAN 77 Interface

Single: `GAMI (A, X)`

Double: The double precision function name is `DGAMI`.

Example

In this example, $\gamma(2.5, 0.9)$ is computed and printed.

```
      USE GAMI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
!                                 Compute
      A     = 2.5
      X     = 0.9
      VALUE = GAMI(A, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' GAMI(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
GAMI( 2.500, 0.900) = 0.1647
```

Description

The incomplete gamma function is defined to be

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt \qquad \text{for } a > 0 \text{ and } x \geq 0$$

The function $\gamma(a, x)$ is defined only for a greater than zero. Although $\gamma(a, x)$ is well defined for $x > -\infty$, this algorithm does not calculate $\gamma(a, x)$ for negative x. For large a and sufficiently large x, $\gamma(a, x)$ may overflow. $\gamma(a, x)$ is bounded by $\Gamma(a)$, and users may find this bound a useful guide in determining legal values of a.

Because logarithmic variables are used, a slight deterioration of two or three digits of accuracy will occur when `GAMI` is very large or very small.

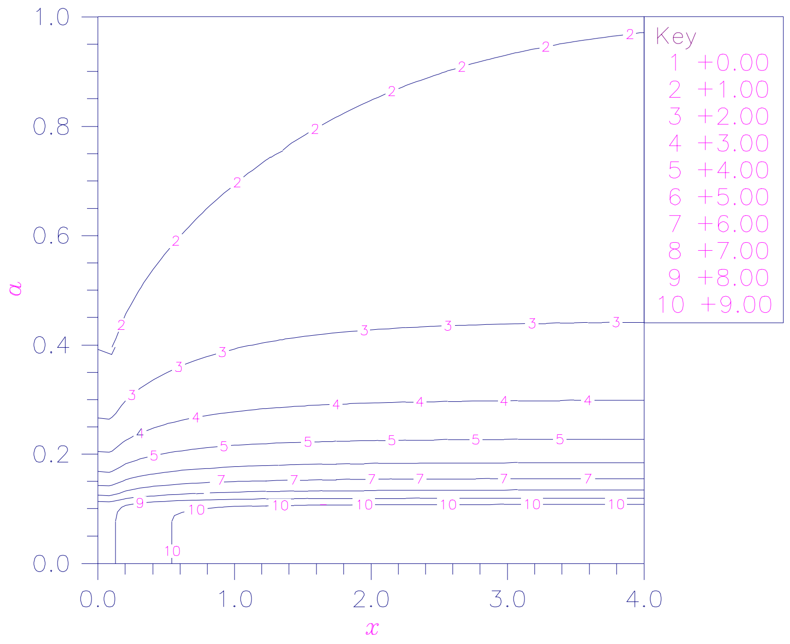


Figure 4-3 Contour Plot of $\gamma(a, x)$

GAMIC

Evaluates the complementary incomplete gamma function.

Function Return Value

GAMIC — Function value. (Output)

Required Arguments

A — The integrand exponent parameter as per the remarks. (Input)

X — The upper limit of the integral definition of `GAMIC`. (Input)
If `A` is positive, then `X` must be positive. Otherwise, `X` must be nonnegative.

FORTRAN 90 Interface

Generic: `GAMIC(A, X)`

Specific: The specific interface names are `S_GAMIC` and `D_GAMIC`.

FORTRAN 77 Interface

Single: `GAMIC(A, X)`

Double: The double precision function name is `DGAMIC`.

Example

In this example, Γ(2.5, 0.9) is computed and printed.

```
      USE GAMIC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
!                                 Compute
      A     = 2.5
      X     = 0.9
      VALUE = GAMIC(A, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' GAMIC(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
GAMIC( 2.500, 0.900) = 1.1646
```

Comments

Informational error

Type	Code	
3	2	Result of `GAMIC(A, X)` is accurate to less than one-half precision because `A` is too near a negative integer.

Description

The incomplete gamma function is defined to be

$$\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$$

The only general restrictions on a are that it must be positive if x is zero; otherwise, it must not be too close to a negative integer such that the accuracy of the result is less than half precision. Furthermore, $\Gamma(a, x)$ must not be so small that it underflows, or so large that it overflows. Although $\Gamma(a, x)$ is well defined for $x > -\infty$ and $a > 0$, this algorithm does not calculate $\Gamma(a, x)$ for negative x.

The function `GAMIC` is based on a code by Gautschi (1979).

GAMIT

This function evaluates the Tricomi form of the incomplete gamma function.

Function Return Value

GAMIT — Function value. (Output)

Required Arguments

A — The integrand exponent parameter as per the comments. (Input)

X — The upper limit of the integral definition of `GAMIT`. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `GAMIT(A, X)`

Specific: The specific interface names are `S_GAMIT` and `D_GAMIT`.

FORTRAN 77 Interface

Single: `GAMIT(A, X)`

Double: The double precision function name is `DGAMIT`.

Example

In this example, $\gamma^*(3.2, 2.1)$ is computed and printed.

```
      USE GAMIT_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
!                                 Compute
      A     = 3.2
      X     = 2.1
      VALUE = GAMIT(A, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' GAMIT(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
GAMIT( 3.200, 2.100) = 0.0284
```

Comments

Informational error

Type	Code	
3	2	Result of `GAMIT(A, X)` is accurate to less than one-half precision because `A` is too close to a negative integer.

Description

The Tricomi's incomplete gamma function is defined to be

$$\gamma^*(a,x) = \frac{x^{-a}\gamma(a,x)}{\Gamma(a)} = \frac{x^{-a}}{\Gamma(a)}\int_x^{\infty} t^{a-1}e^{-t}\,dt$$

where $\gamma(a, x)$ is the incomplete gamma function. See `GAMI` (page 59) for the definition of $\gamma(a, x)$.

The only general restriction on a is that it must not be too close to a negative integer such that the accuracy of the result is less than half precision. Furthermore, $|\gamma^*(a, x)|$ must not underflow or overflow. Although $\gamma^*(a, x)$ is well defined for $x > -\infty$, this algorithm does not calculate $\gamma^*(a, x)$ for negative x.

A slight deterioration of two or three digits of accuracy will occur when `GAMIT` is very large or very small in absolute value because logarithmic variables are used. Also, if the parameter a is very close to a negative integer (but not quite a negative integer), there is a loss of accuracy which is reported if the result is less than half machine precision.

The function `GAMIT` is based on a code by Gautschi (1979).

PSI

This function evaluates the logarithmic derivative of the gamma function.

Function Return Value

PSI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `PSI (X)`

Specific: The specific interface names are `S_PSI`, `D_PSI`, and `C_PSI`.

FORTRAN 77 Interface

Single: `PSI (X)`

Double:	The double precision function name is `DPSI`.
Complex:	The complex name is `CPSI`.

Example

In this example, $\psi(1.915)$ is computed and printed.

```
      USE PSI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.915
      VALUE = PSI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' PSI(', F6.3, ') = ', F6.3)
      END
```

Output

```
PSI( 1.915) = 0.366
```

Comments

Informational error

Type	Code	
3	2	Result of `PSI`(`X`) is accurate to less than one-half precision because `X` is too near a negative integer.

Description

The psi function, also called the digamma function, is defined to be

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

See `GAMMA` (page 51) for the definition of $\Gamma(x)$.

The argument x must not be exactly zero or a negative integer, or $\psi(x)$ is undefined. Also, x must not be too close to a negative integer such that the accuracy of the result is less than half precision.

Additional Example

In this example, $\psi(1.9 + 4.3i)$ is computed and printed.

```
      USE PSI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.9, 4.3)
      VALUE = PSI(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' PSI(', F6.3, ',', F6.3, ') = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
PSI( 1.900, 4.300) = ( 1.507, 1.255)
```

POCH

This function evaluates a generalization of Pochhammer's symbol.

Function Return Value

POCH — Function value. (Output)
The generalized Pochhammer symbol is $\Gamma(a + x)/\Gamma(a)$.

Required Arguments

A — The first argument. (Input)

X — The second, differential argument. (Input)

FORTRAN 90 Interface

Generic: `POCH (A, X)`

Specific: The specific interface names are `S_POCH` and `D_POCH`.

FORTRAN 77 Interface

Single: `POCH (A, X)`

Double: The double precision function name is `DPOCH`.

Example

In this example, $(1.6)_{0.8}$ is computed and printed.

```
      USE POCH_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
 !                                 Compute
```

```
      A     = 1.6
      X     = 0.8
      VALUE = POCH(A, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' POCH(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
POCH( 1.600, 0.800) = 1.3902
```

Comments

1. Informational errors

Type	Code	
3	2	Result of `POCH(A, X)` is accurate to less than one-half precision because the absolute value of the `X` is too large. Therefore, `A` + `X` cannot be evaluated accurately.
3	2	Result of `POCH(A, X)` is accurate to less than one-half precision because either `A` or `A` + `X` is too close to a negative integer.

2. For `X` a nonnegative integer, `POCH(A, X)` is just Pochhammer's symbol.

Description

Pochhammer's symbol is $(a)_n = (a)(a - 1)\ldots(a - n + 1)$ for n a nonnegative integer. Pochhammer's generalized symbol is defined to be

$$(a)_x = \frac{\Gamma(a+x)}{\Gamma(a)}$$

See `GAMMA` (page 51) for the definition of $\Gamma(x)$.

Note that a straightforward evaluation of Pochhammer's generalized symbol with either gamma or log gamma functions can be especially unreliable when a is large or x is small.

Substantial loss can occur if $a + x$ or a are close to a negative integer unless $|x|$ is sufficiently small. To insure that the result does not overflow or underflow, one can keep the arguments a and $a + x$ well within the range dictated by the gamma function routine `GAMMA` or one can keep $|x|$ small whenever a is large. `POCH` also works for a variety of arguments outside these rough limits, but any more general limits that are also useful are difficult to specify.

POCH1

This function evaluates a generalization of Pochhammer's symbol starting from the first order.

Function Return Value

POCH1 — Function value. (Output)
POCH1(A, X) = (POCH(A, X) – 1)/X.

Required Arguments

A — The first argument. (Input)

X — The second, differential argument. (Input)

FORTRAN 90 Interface

Generic: POCH1 (A, X)

Specific: The specific interface names are S_POCH1 and D_POCH1.

FORTRAN 77 Interface

Single: POCH1 (A, X)

Double: The double precision function name is DPOCH1.

Example

In this example, POCH1(1.6, 0.8) is computed and printed.

```
      USE POCH1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
!                                 Compute
      A     = 1.6
      X     = 0.8
      VALUE = POCH1(A, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' POCH1(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
POCH1( 1.600, 0.800) = 0.4878
```

Description

Pochhammer's symbol from the first order is defined to be

$$\mathrm{POCH1}(a, x) = \frac{(a)_x - 1}{x} = \frac{\Gamma(a+x)}{\Gamma(a) - 1} / x$$

where $(a)_x$ is Pochhammer's generalized symbol. See POCH (page 66) for the definition of $(a)_x$. It is useful in special situations that require especially accurate values when x is small. This specification is particularly suited for stability when computing expressions such as

$$\left[\frac{\Gamma(a+x)}{\Gamma(a)} - \frac{\Gamma(b+x)}{\Gamma(b)}\right] / x = \mathrm{POCH1}(a,x) - \mathrm{POCH1}(b,x)$$

Note that $\mathrm{POCH1}(a, 0) = \psi(a)$. See PSI (page 64) for the definition of $\psi(a)$.

When $|x|$ is so small that substantial cancellation will occur if the straightforward formula is used, we use an expansion due to fields and discussed by Luke (1969).

The ratio $(a)_x = \Gamma(a + x)/\Gamma(a)$ is written by Luke as $(a + (x - 1)/2)^x$ times a polynomial in $(a + (x - 1)/2)^{-2}$. To maintain significance in POCH1, we write for positive a.

$$(a + (x - 1)/2)^x = \exp(x \ln(a + (x - 1)/2)) = e^q = 1 + q\mathrm{EXPRL}(q)$$

where $\mathrm{EXPRL} = (e^x - 1)/x$. Likewise, the polynomial is written $P = 1 + xP_1(a, x)$. Thus,

$$\mathrm{POCH1}\ (a, x) = ((a)_x - 1)/\mathrm{x} = \mathrm{EXPRL}(q)(q/x + qP_1(a, x)) + P_1(a, x)$$

Substantial significance loss can occur if $a + x$ or a are close to a negative integer even when $|x|$ is very small. To insure that the result does not overflow or underflow, one can keep the arguments a and $a + x$ well within the range dictated by the gamma function routine GAMMA (page 51) or one can keep $|x|$ small whenever a is large. POCH also works for a variety of arguments outside these rough limits, but any more general limits that are also useful are difficult to specify.

BETA

This function evaluates the complete beta function.

Function Return Value

BETA — Function value. (Output)

Required Arguments

A — First beta parameter. (Input)
For real arguments, A must be positive.

B — Second beta parameter. (Input)
For real arguments, B must be positive.

FORTRAN 90 Interface

Generic: `BETA (A, B)`

Specific: The specific interface names are `S_BETA`, `D_BETA`, and `C_BETA`.

FORTRAN 77 Interface

Single:	`BETA(A, B)`
Double:	The double precision function name is `DBETA`.
Complex:	The complex name is `CBETA`.

Example

In this example, β(2.2, 3.7) is computed and printed.

```
      USE BETA_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
!                                 Compute
      A     = 2.2
      X     = 3.7
      VALUE = BETA(A, X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' BETA(', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
BETA( 2.200, 3.700) = 0.0454
```

Comments

Informational error

Type	Code	
2	1	The function underflows because `A` and/or `B` is too large.

Description

The beta function is defined to be

$$\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}(1-t)^{b-1}dt$$

See `GAMMA` (page 51) for the definition of $\Gamma(x)$.

For real arguments the function `BETA` requires that both arguments be positive. In addition, the arguments must not be so large that the result underflows.

For complex arguments, the arguments a and $a + b$ must not be close to negative integers. The arguments should not be so large (near the real axis) that the result underflows. Also, $a + b$ should not be so far from the real axis that the result overflows.

Additional Example

In this example, β(1.7 + 2.2*i*, 3.7 + 0.4*i*) is computed and printed.

```
      USE BETA_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    A, B, VALUE
!                                 Compute
      A     = (1.7, 2.2)
      B     = (3.7, 0.4)
      VALUE = BETA(A, B)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, B, VALUE
99999 FORMAT (' BETA((', F6.3, ',', F6.3, '), (', F6.3, ',', F6.3,&
           ')) = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
BETA(( 1.700, 2.200), ( 3.700, 0.400)) = (-0.033,-0.017)
```

ALBETA

This function evaluates the natural logarithm of the complete beta function for positive arguments.

Function Return Value

ALBETA — Function value. (Output)
`ALBETA` returns ln β(`A`, `B`) = ln(Γ(`A`)Γ(`B`))/Γ(`A` + `B`).

Required Arguments

A — The first argument of the `BETA` function. (Input)
For real arguments, `A` must be greater than zero.

B — The second argument of the `BETA` function. (Input)
For real arguments, `B` must be greater than zero.

FORTRAN 90 Interface

Generic: `ALBETA (A, B)`

Specific: The specific interface names are `S_ALBETA`, `D_ALBETA`, and `C_ALBETA`.

FORTRAN 77 Interface

Single: `ALBETA (A, B)`

Double: The double precision function name is DLBETA.

Complex: The complex name is CLBETA.

Example

In this example, ln β(2.2, 3.7) is computed and printed.

```
      USE ALBETA_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       A, VALUE, X
!                                  Compute
      A     = 2.2
      X     = 3.7
      VALUE = ALBETA(A, X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, X, VALUE
99999 FORMAT (' ALBETA(', F6.3, ',', F6.3, ') = ', F8.4)
      END
```

Output

```
ALBETA( 2.200, 3.700) = -3.0928
```

Comments

Note that ln β(A, B) = ln β(B, A).

Description

ALBETA computes ln β(*a*, *b*) = ln β(*b*, *a*). See BETA (page 69) for the definition of β(*a*, *b*).

For real arguments, the function ALBETA is defined for $a > 0$ and $b > 0$. It returns accurate results even when *a* or *b* is very small. It can overflow for very large arguments; this error condition is not detected except by the computer hardware.

For complex arguments, the arguments *a*, *b* and $a + b$ must not be close to negative integers (even though some combinations ought to be allowed). The arguments should not be so large that the logarithm of the gamma function overflows (presumably an improbable condition).

Additional Example

In this example, ln β(1.7 + 2.2*i*, 3.7 + 0.4*i*) is computed and printed.

```
      USE ALBETA_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      COMPLEX    A, B, VALUE
!                                  Compute
      A     = (1.7, 2.2)
      B     = (3.7, 0.4)
      VALUE = ALBETA(A, B)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) A, B, VALUE
99999 FORMAT (' ALBETA((', F6.3, ',', F6.3, '), (', F6.3, ',', F6.3, &
            ')) = (', F6.3, ',', F6.3, ')')
      END
```

Output

```
ALBETA(( 1.700, 2.200), ( 3.700, 0.400)) = (-3.280,-2.659)
```

BETAI

This function evaluates the incomplete beta function ratio.

Function Return Value

BETAI — Probability that a random variable from a beta distribution having parameters `PIN` and `QIN` will be less than or equal to `X`. (Output)

Required Arguments

X — Upper limit of integration. (Input)
`X` must be in the interval (0.0, 1.0) inclusive.

PIN — First beta distribution parameter. (Input)
`PIN` must be positive.

QIN — Second beta distribution parameter. (Input)
`QIN` must be positive.

FORTRAN 90 Interface

Generic: `BETAI(X, PIN, QIN)`

Specific: The specific interface names are `S_BETAI` and `D_BETAI`.

FORTRAN 77 Interface

Single: `BETAI(X, PIN, QIN)`

Double: The double precision function name is `DBETAI`.

Example

In this example, $I_{0.61}(2.2, 3.7)$ is computed and printed.

```
      USE BETAI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       PIN, QIN, VALUE, X
!                                 Compute
      X     = 0.61
      PIN   = 2.2
      QIN   = 3.7
      VALUE = BETAI(X, PIN, QIN)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, PIN, QIN, VALUE
99999 FORMAT (' BETAI(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.4)
      END
```

Output

```
 BETAI( 0.610, 2.200, 3.700) = 0.8822
```

Description

The incomplete beta function is defined to be

$$I_x(p, q) = \frac{\beta_x(p, q)}{\beta(p, q)} = \frac{1}{\beta(p, q)} \int_0^x t^{p-1}(1-t)^{q-1} dt$$
$$\text{for } 0 \le x \le 1, p > 0, q > 0$$

See BETA (page 69) for the definition of β(*p*, *q*).

The parameters *p* and *q* must both be greater than zero. The argument *x* must lie in the range 0 to 1. The incomplete beta function can underflow for sufficiently small *x* and large *p*; however, this underflow is not reported as an error. Instead, the value zero is returned as the function value.

The function BETAI is based on the work of Bosten and Battiste (1974).

Chapter 5: Error Function and Related Functions

Routines

Usage Notes

The error function is

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

The complementary error function is erfc(*x*) = 1 − erf(*x*). Dawson's function is

$$e^{-x^2} \int_0^x e^{t^2} dt$$

The Fresnel integrals are

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2} t^2\right) dt$$

and

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

They are related to the error function by

$$C(z) + iS(z) = \frac{1+i}{2}\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}(1-i)z\right)$$

ERF

This function evaluates the error function.

Function Return Value

ERF — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `ERF (X)`

Specific: The specific interface names are `S_ERF` and `D_ERF`.

FORTRAN 77 Interface

Single: `ERF (X)`

Double: The double precision function name is `DERF`.

Example

In this example, erf(1.0) is computed and printed.

```
      USE ERF_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.0
      VALUE = ERF(X)
```

```
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERF(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERF( 1.000) = 0.843
```

Description

The error function, erf(x), is defined to be

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

All values of x are legal.

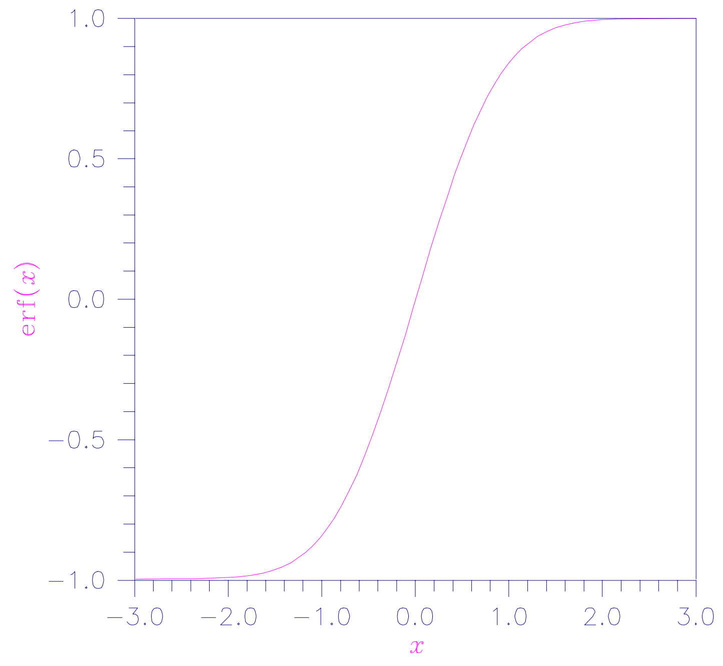


Figure 5-1 Plot of erf x

ERFC

This function evaluates the complementary error function.

Function Return Value

ERFC — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic:	`ERFC(X)`
Specific:	The specific interface names are `S_ERFC` and `D_ERFC`.

FORTRAN 77 Interface

Single:	`ERFC(X)`
Double:	The double precision function name is `DERFC`.

Example

In this example, erfc(1.0) is computed and printed.

```
      USE ERFC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.0
      VALUE = ERFC(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFC(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFC( 1.000) = 0.157
```

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Description

The complementary error function, erfc(x), is defined to be

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

The argument x must not be so large that the result underflows. Approximately, x should be less than

$$\left[-\ln\left(\sqrt{\pi}s\right)\right]^{1/2}$$

where s = AMACH(1) (see the Reference Material section of this manual) is the smallest representable positive floating-point number.

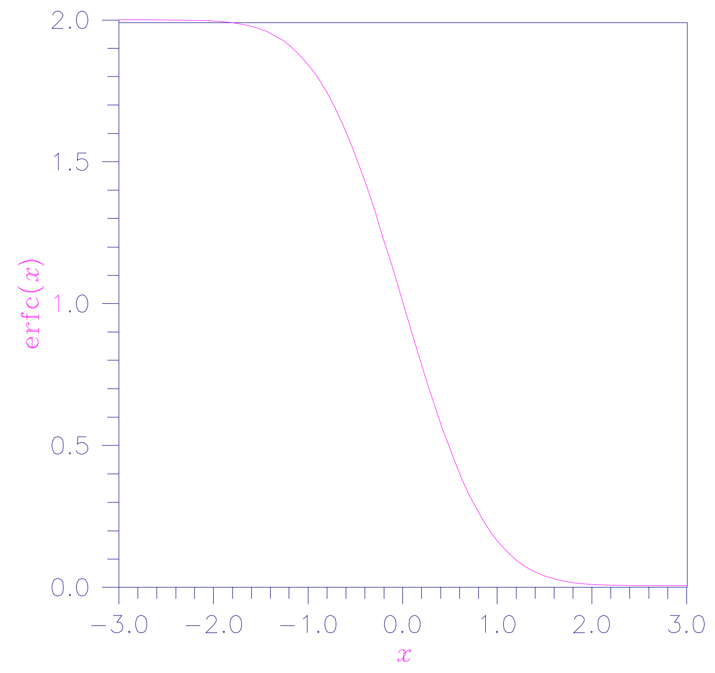


Figure 5-2 Plot of erfc x

ERFCE

This function evaluates the exponentially scaled complementary error function.

Function Return Value

ERFCE — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `ERFCE (X)`

Specific: The specific interface names are `S_ERFCE` and `D_ERFCE`.

FORTRAN 77 Interface

Single:	`ERFCE (X)`
Double:	The double precision function name is `DERFCE`.

Example

In this example, `ERFCE`(1.0) = $e^{1.0}$ erfc(1.0) is computed and printed.

```
      USE ERFCE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.0
      VALUE = ERFCE(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFCE(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFCE( 1.000) = 0.428
```

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Description

The function `ERFCE`(`X`) computes

$$e^{x^2} \text{ erfc } (x)$$

where erfc(x) is the complementary error function. See `ERFC` (page 77) for its definition.

To prevent the answer from underflowing, x must be greater than

$$x_{\min} \simeq -\sqrt{\ln(b/2)}$$

where b = `AMACH`(2) is the largest representable floating-point number.

CERFE

This function evaluates a scaled function related to `ERFC`..

Function Return Value

CERFE — Complex function value. (Output)

Required Arguments

Z — Complex argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `CERFE (Z)`

Specific: The specific interface names are `C_CERFE` and `Z_CERFE`.

FORTRAN 77 Interface

Complex: `CERFE (Z)`

Double complex: The double complex function name is `ZERFE`.

Example

In this example, `CERFE`$(2.5 + 2.5i)$ is computed and printed.

```
      USE CERFE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (2.5, 2.5)
      VALUE = CERFE(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CERFE(', F6.3, ',', F6.3, ') = (', &
            F6.3, ',', F6.3, ')')
      END
```

Output

```
CERFE( 2.500, 2.500) = ( 0.117, 0.108)
```

Description

Function `CERFCE` is defined to be

$$e^{-z^2}\operatorname{erfc}(-iz) = -ie^{-z^2}\frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{t^2}\,dt$$

Let b = `AMACH`(2) be the largest floating-point number. The argument z must satisfy

$$|z| \le \sqrt{b}$$

or else the value returned is zero. If the argument z does not satisfy $(\Im z)^2 - (\Re z)^2 \le \log b$, then b is returned. All other arguments are legal (Gautschi 1969, 1970).

ERFI

This function evaluates the inverse error function.

Function Return Value

ERFI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `ERFI (X)`

Specific: The specific interface names are `S_ERFI` and `D_ERFI`.

FORTRAN 77 Interface

Single: `ERFI (X)`

Double: The double precision function name is `DERFI`.

Example

In this example, $\text{erf}^{-1}(\text{erf}(1.0))$ is computed and printed.

```
      USE ERFI_INT
      USE ERF_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = ERF(1.0)
      VALUE = ERFI(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFI(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFI( 0.843) = 1.000
```

Comments

Informational error

Type	Code	
3	2	Result of `ERFI(X)` is accurate to less than one-half precision because the absolute value of the argument is too large.

Description

Function `ERFI(X)` computes the inverse of the error function erf x, defined in `ERF` (page 76).

The function `ERFI(X)` is defined for $|x| < 1$. If $x_{\max} < |x| < 1$, then the answer will be less accurate than half precision. Very approximately,

$$x_{\max} \approx 1 - \sqrt{\varepsilon/(4\pi)}$$

where ε = `AMACH`(4) is the machine precision.

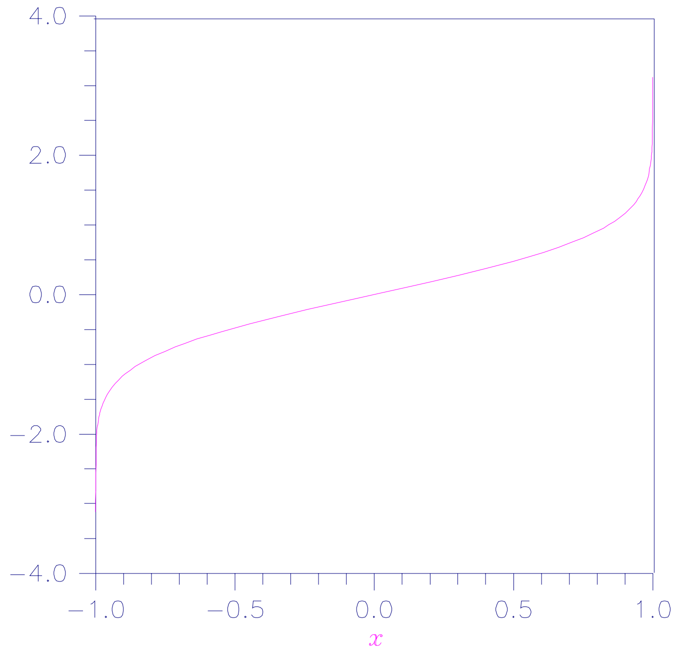


Figure 5-3 Plot of $\mathrm{erf}^{-1}x$

ERFCI

This function evaluates the inverse complementary error function.

Function Return Value

ERFCI — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `ERFCI (X)`

Specific: The specific interface names are `S_ERFCI` and `D_ERFCI`.

FORTRAN 77 Interface

Single: `ERFCI` (X)

Double: The double precision function name is `DERFCI`.

Example

In this example, $\mathrm{erfc}^{-1}(\mathrm{erfc}(1.0))$ is computed and printed.

```
      USE ERFCI_INT
      USE ERFC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = ERFC(1.0)
      VALUE = ERFCI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ERFCI(', F6.3, ') = ', F6.3)
      END
```

Output

```
ERFCI( 0.157) = 1.000
```

Comments

Informational error

Type	Code	
3	2	Result of `ERFCI(X)` is accurate to less than one-half precision because the argument is too close to 2.0.

Description

The function `ERFCI(X)` computes the inverse of the complementary error function erfc *x*, defined in `ERFC` (page 77).

The function ERFCI(X) is defined for $0 < x < 2$. If $x_{\max} < x < 2$, then the answer will be less accurate than half precision. Very approximately,

$$x_{\max} \approx 2 - \sqrt{\varepsilon/(4\pi)}$$

where ε = AMACH(4) is the machine precision.

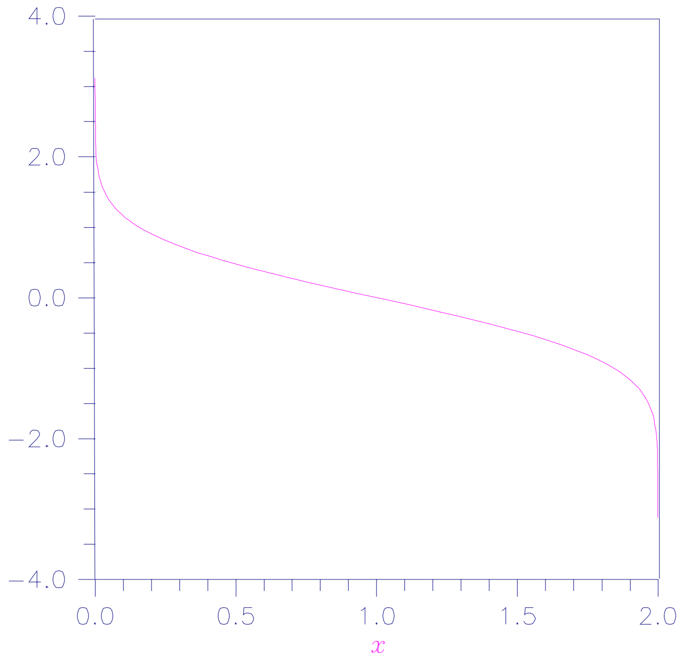


Figure 5-4   Plot of $\mathrm{erf}^{-1}x$

DAWS

This function evaluates Dawson's function.

Function Return Value

DAWS — Function value.   (Output)

Required Arguments

X — Argument for which the function value is desired.   (Input)

FORTRAN 90 Interface

Generic: DAWS (X)

Specific: The specific interface names are S_DAWS and D_DAWS.

FORTRAN 77 Interface

Single: `DAWS (X)`

Double: The double precision function name is `DDAWS`.

Example

In this example, `DAWS`(1.0) is computed and printed.

```
      USE DAWS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.0
      VALUE = DAWS(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' DAWS(', F6.3, ') = ', F6.3)
      END
```

Output

```
DAWS( 1.000) = 0.538
```

Comments

1. Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too large.

2. The Dawson function is closely related to the error function for imaginary arguments.

Description

Dawson's function is defined to be

$$e^{-x^2}\int_0^x e^{t^2}dt$$

It is closely related to the error function for imaginary arguments.

So that Dawson's function does not underflow, $|x|$ must be less than $1/(2s)$. Here, s = `AMACH`(1) is the smallest representable positive floating-point number.

FRESC

This function evaluates the cosine Fresnel integral.

Function Return Value

FRESC — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `FRESC (X)`

Specific: The specific interface names are `S_FRESC` and `D_FRESC`.

FORTRAN 77 Interface

Single: `FRESC (X)`

Double: The double precision function name is `DFRESC`.

Example

In this example, $C(1.75)$ is computed and printed.

```
      USE FRESC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.75
      VALUE = FRESC(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' FRESC(', F6.3, ') = ', F6.3)
      END
```

Output

```
FRESC( 1.750) = 0.322
```

Description

The cosine Fresnel integral is defined to be

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt$$

All values of x are legal.

FRESS

This function evaluates the sine Fresnel integral.

Function Value Return

FRESS — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `FRESS (X)`

Specific: The specific interface names are `S_FRESS` and `D_FRESS`.

FORTRAN 77 Interface

Single: `FRESS (X)`

Double: The double precision function name is `DFRESS`.

Example

In this example, $S(1.75)$ is computed and printed.

```
      USE FRESS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 1.75
      VALUE = FRESS(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' FRESS(', F6.3, ') = ', F6.3)
      END
```

Output

```
FRESS( 1.750) = 0.499
```

Description

The sine Fresnel integral is defined to be

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right) dt$$

All values of *x* are legal.

Chapter 6: Bessel Functions

Routines

Usage Notes

The following table lists the Bessel function routines by argument and order type:

	Real Argument				Complex Argument	
	Order				Order	
Function	0	1	integer	real	Integer	Real
$J_\nu(x)$	`BSJ0` p. 92	`BSJ1` p. 94	`BSJNS` p. 109	`BSJS` p. 113	`BSJNS` p. 109	`CBJS` p. 123
$Y_\nu(x)$	`BSY0` p. 95	`BSY1` p. 97		`BSYS` p. 115		`CBYS` p. 125
$I_\nu(x)$	`BSI0` p. 98	`BSI1` p. 100	`BSINS` p. 111	`BSIS` p. 117	`BSINS` p. 111	`CBIS` p. 127
$e^{-\lvert x\rvert}I_\nu(x)$	`BSI0E` p. 104	`BSI1E` p. 106		`BSIES` p. 118		
$K_\nu(x)$	`BSK0` p. 101	`BSK1` p. 103		`BSKS` p. 120		`CBKS` p. 129
$e^{-\lvert x\rvert}K_\nu(x)$	`BSK0E` p. 107	`BSK1E` p. 108		`BSKES` p. 121		

BSJ0

This function evaluates the Bessel function of the first kind of order zero.

Function Value Return

BSJ0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSJ0 (X)`

Specific: The specific interface names are `S_BSJ0` and `D_BSJ0`.

FORTRAN 77 Interface

Single: `BSJ0 (X)`

Double: The double precision function name is DBSJ0.

Example

In this example, $J_0(3.0)$ is computed and printed.

```
      USE BSJ0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 3.0
      VALUE = BSJ0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSJ0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSJ0( 3.000) = -0.260
```

Description

The Bessel function $J_0(x)$ is defined to be

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta$$

To prevent the answer from being less accurate than half precision, $|x|$ should be smaller than

$$1/\sqrt{\varepsilon}$$

For the result to have any precision at all, $|x|$ must be less than $1/\varepsilon$. Here, ε is the machine precision, ε = AMACH(4).

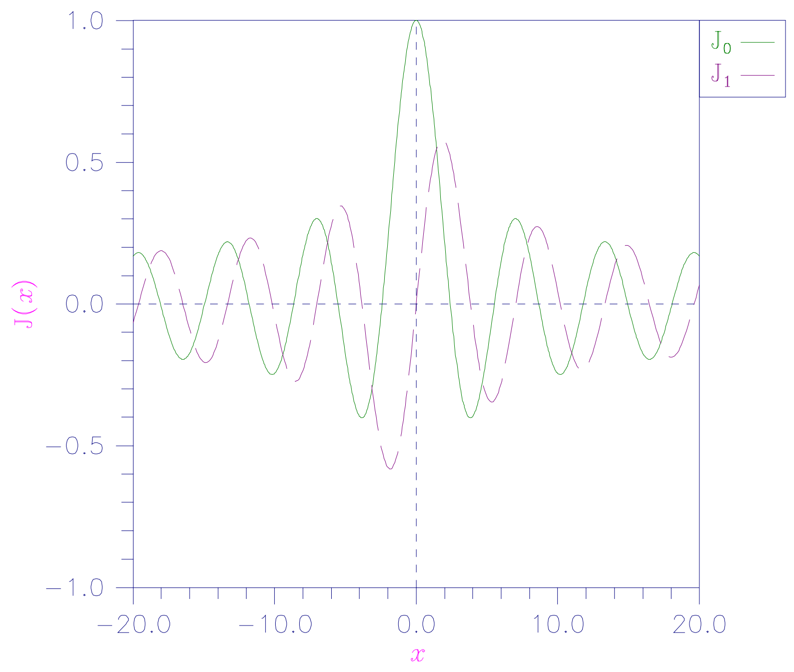


Figure 6-1   Plot of $J_0(x)$ and $J_1(x)$

BSJ1

This function evaluates the Bessel function of the first kind of order one.

Function Return Value

BSJ1 — Function value.   (Output)

Required Arguments

X — Argument for which the function value is desired.   (Input)

FORTRAN 90 Interface

Generic: `BSJ1 (X)`

Specific: The specific interface names are `S_BSJ1` and `D_BSJ1`.

FORTRAN 77 Interface

Single: `BSJ1 (X)`

Double: The double precision function name is `DBSJ1`.

Example

In this example, $J_1(2.5)$ is computed and printed.

```
      USE BSJ1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 2.5
      VALUE = BSJ1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSJ1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSJ1( 2.500) = 0.497
```

Comments

Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too small.

Description

The Bessel function $J_1(x)$ is defined to be

$$J_1(x) = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin \theta - \theta) \, d\theta$$

The argument x must be zero or larger in absolute value than $2s$ to prevent $J_1(x)$ from underflowing. Also, $|x|$ should be smaller than

$$1/\sqrt{\varepsilon}$$

to prevent the answer from being less accurate than half precision. $|x|$ must be less than $1/\varepsilon$ for the result to have any precision at all. Here, ε is the machine precision, ε = `AMACH`(4), and s = `AMACH`(1) is the smallest representable positive floating-point number.

BSY0

This function evaluates the Bessel function of the second kind of order zero.

Function Return Value

BSY0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSY0 (X)`

Specific: The specific interface names are `S_BSY0` and `D_BSY0`.

FORTRAN 77 Interface

Single: `BSY0 (X)`

Double: The double precision function name is `DBSY0`.

Example

In this example, $Y_0(3.0)$ is computed and printed.

```
      USE BSY0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 3.0
      VALUE = BSY0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSY0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSY0( 3.000) = 0.377
```

Description

The Bessel function $Y_0(x)$ is defined to be

$$Y_0(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x \sin\theta)\, d\theta - \frac{2}{\pi}\int_0^{\infty} e^{-z\sinh t}\, dt$$

To prevent the answer from being less accurate than half precision, x should be smaller than

$$1/\sqrt{\varepsilon}$$

For the result to have any precision at all, $|x|$ must be less than $1/\varepsilon$. Here, ε is the machine precision, ε = AMACH(4).

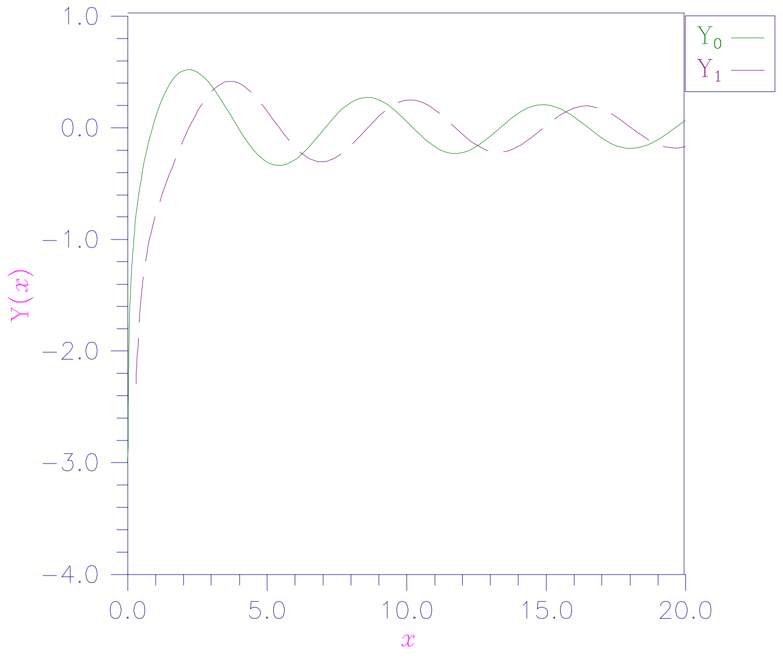


Figure 6-2 Plot of $Y_0(x)$ and $Y_1(x)$

BSY1

This function evaluates the Bessel function of the second kind of order one.

Function Return Value

BSY1 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSY1 (X)`

Specific: The specific interface names are `S_BSY1` and `D_BSY1`.

FORTRAN 77 Interface

Single: `BSY1 (X)`

Double: The double precision function name is DBSY1.

Example

In this example, $Y_1(3.0)$ is computed and printed.

```
      USE BSY1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 3.0
      VALUE = BSY1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSY1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSY1( 3.000) = 0.325
```

Description

The Bessel function $Y_1(x)$ is defined to be

$$Y_1(x) = -\frac{1}{\pi}\int_0^{\pi} \sin(\theta - x\sin\theta)\, d\theta - \frac{1}{\pi}\int_0^{\infty}\left\{e^t - e^{-t}\right\}e^{-z\sinh t}dt$$

$Y_1(x)$ is defined for $x > 0$. To prevent the answer from being less accurate than half precision, x should be smaller than

$$1/\sqrt{\varepsilon}$$

For the result to have any precision at all, $|x|$ must be less than $1/\varepsilon$. Here, ε is the machine precision, ε = AMACH(4).

BSI0

This function evaluates the modified Bessel function of the first kind of order zero.

Function Return Value

BSI0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSI0 (X)`

Specific: The specific interface names are `S_BSI0` and `D_BSI0`.

FORTRAN 77 Interface

Single: `BSI0 (X)`

Double: The double precision function name is `DBSI0`.

Example

In this example, $I_0(4.5)$ is computed and printed.

```
      USE BSI0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 4.5
      VALUE = BSI0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI0( 4.500) = 17.481
```

Description

The Bessel function $I_0(x)$ is defined to be

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} \cosh(x\cos\theta)\,d\theta$$

The absolute value of the argument x must not be so large that $e^{|x|}$ overflows.

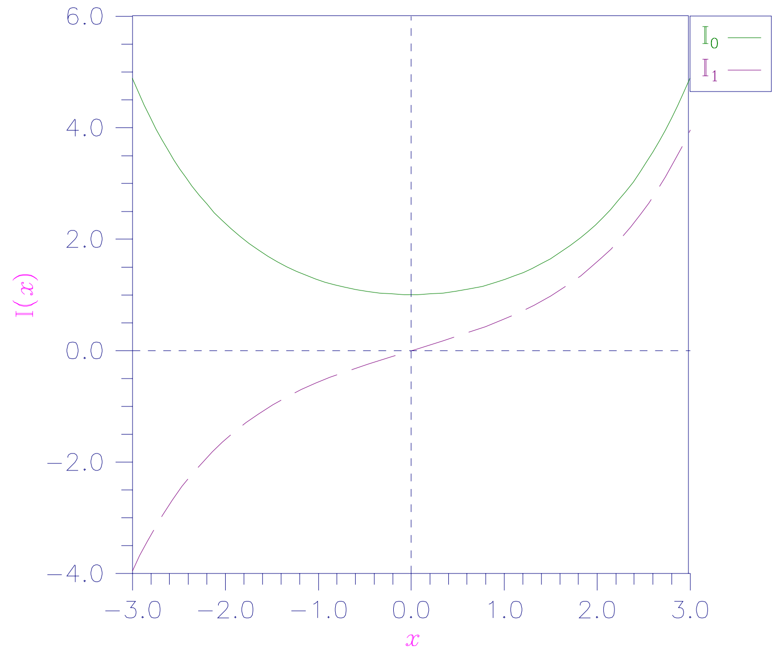


Figure 6-3 Plot of $I_0(x)$ and $I_1(x)$

BSI1

This function evaluates the modified Bessel function of the first kind of order one.

Function Return Value

BSI1 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSI1 (X)`

Specific: The specific interface names are `S_BSI1` and `D_BSI1`.

FORTRAN 77 Interface

Single: `BSI1 (X)`

Double: The double precision function name is `DBSI1`.

Example

In this example, $I_1(4.5)$ is computed and printed.

```
      USE BSI1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 4.5
      VALUE = BSI1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI1( 4.500) = 15.389
```

Comments

Informational error

Type	Code	
2	1	The function underflows because the absolute value of `X` is too small.

Description

The Bessel function $I_1(x)$ is defined to be

$$I_1(x) = \frac{1}{\pi} \int_0^{\pi} e^{x \cos\theta} \cos\theta \, d\theta$$

The argument should not be so close to zero that $I_1(x) \approx x/2$ underflows, nor so large in absolute value that $e^{|x|}$ and, therefore, $I_1(x)$ overflows.

BSK0

This function evaluates the modified Bessel function of the second kind of order zero.

Function Return Value

BSK0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSK0 (X)`

Specific: The specific interface names are `S_BSK0` and `D_BSK0`.

FORTRAN 77 Interface

Single: `BSK0 (X)`

Double: The double precision function name is `DBSK0`.

Example

In this example, $K_0(0.5)$ is computed and printed.

```
      USE BSK0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.5
      VALUE = BSK0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK0( 0.500) = 0.924
```

Comments

Informational error

Type	Code	
2	1	The function underflows because `X` is too large.

Description

The Bessel function $K_0(x)$ is defined to be

$$K_0(x) = \int_0^\infty \cos(x \sinh t)\, dt$$

The argument must be larger than zero, but not so large that the result, approximately equal to

$$\sqrt{\pi/(2x)}\, e^{-x}$$

underflows.

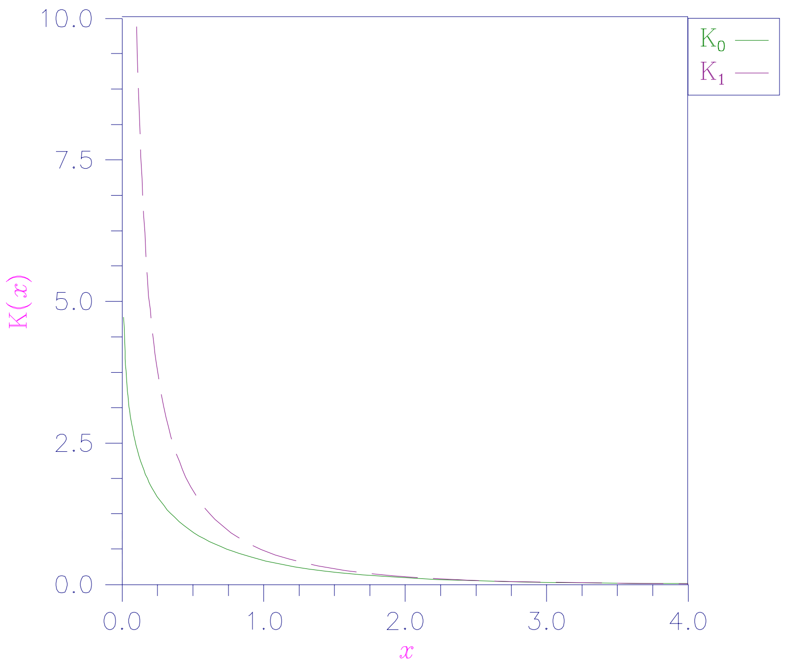


Figure 6-4   Plot of $K_0(x)$ and $K_1(x)$

BSK1

This function evaluates the modified Bessel function of the second kind of order one.

Function Return Value

BSK1 — Function value.   (Output)

Required Arguments

X — Argument for which the function value is desired.   (Input)

FORTRAN 90 Interface

Generic: `BSK1 (X)`

Specific: The specific interface names are `S_BSK1` and `D_BSK1`.

FORTRAN 77 Interface

Single: `BSK1 (X)`

Double: The double precision function name is `DBSK1`.

Example

In this example, $K_1(0.5)$ is computed and printed.

```
      USE BSK1_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = 0.5
      VALUE = BSK1(X)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK1( 0.500) = 1.656
```

Comments

Informational error

Type	Code	
2	1	The function underflows because X is too large.

Description

The Bessel function $K_1(x)$ is defined to be

$$K_1(x) = \int_0^\infty \sin(x \sinh t) \sinh t \, dt$$

The argument x must be large enough (> max(1/b, s)) that $K_1(x)$ does not overflow, and x must be small enough that the approximate answer,

$$\sqrt{\pi/(2x)}\, e^{-x}$$

does not underflow. Here, s is the smallest representable positive floating-point number, s = `AMACH`(1) , and b = `AMACH`(2) is the largest representable floating-point number.

BSI0E

This function evaluates the exponentially scaled modified Bessel function of the first kind of order zero.

Function Return Value

BSI0E — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSI0E (X)`

Specific: The specific interface names are `S_BSI0E` and `D_BSI0E`.

FORTRAN 77 Interface

Single: `BSI0E (X)`

Double: The double precision function name is `DBSI0E`.

Example

In this example, `BSI0E`(4.5) is computed and printed.

```
      USE BSI0E_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 4.5
      VALUE = BSI0E(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI0E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI0E( 4.500) = 0.194
```

Description

Function `BSI0E` computes $e^{-|x|} I_0(x)$. For the definition of the Bessel function $I_0(x)$, see `BSI0` (page 98).

BSI1E

This function evaluates the exponentially scaled modified Bessel function of the first kind of order one.

Function Return Value

BSI1E — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSI1E (X)`

Specific: The specific interface names are `S_BSI1E` and `D_BSI1E`.

FORTRAN 77 Interface

Single: `BSI1E (X)`

Double: The double precision function name is `DBSI1E`.

Example

In this example, `BSI1E`(4.5) is computed and printed.

```
      USE BSI1E_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 4.5
      VALUE = BSI1E(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSI1E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSI1E( 4.500) = 0.171
```

Comments

Informational error
Type   Code

2	1	The function underflows because the absolute value of `X` is too small.

Description

Function `BSI1E` computes $e^{-|x|} I_1(x)$. For the definition of the Bessel function $I_1(x)$, see `BSI1` (page 100). The function `BSI1E` underflows if $|x|/2$ underflows.

BSK0E

This function evaluates the exponentially scaled modified Bessel function of the second kind of order zero.

Function Return Value

BSK0E — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSK0E (X)`

Specific: The specific interface names are `S_BSK0E` and `D_BSK0E`.

FORTRAN 77 Interface

Single: `BSK0E (X)`

Double: The double precision function name is `DBSK0E`.

Example

In this example, `BSK0E`(0.5) is computed and printed.

```
      USE BSK0E_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.5
      VALUE = BSK0E(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK0E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK0E( 0.500) = 1.524
```

Description

Function `BSK0E` computes $e^x K_0(x)$. For the definition of the Bessel function $K_0(x)$, see `BSK0` (page 101). The argument must be greater than zero for the result to be defined.

BSK1E

This function evaluates the exponentially scaled modified Bessel function of the second kind of order one.

Function Return Value

BSK1E — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BSK1E (X)`

Specific: The specific interface names are `S_BSK1E` and `D_BSK1E`.

FORTRAN 77 Interface

Single: `BSK1E (X)`

Double: The double precision function name is `DBSK1E`.

Example

In this example, `BSK1E`(0.5) is computed and printed.

```
      USE BSK1E_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.5
      VALUE = BSK1E(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BSK1E(', F6.3, ') = ', F6.3)
      END
```

Output

```
BSK1E( 0.500) = 2.731
```

Description

Function `BSK1E` computes $e^x K_1(x)$. For the definition of the Bessel function $K_1(x)$, see `BSK1` (page 103). The answer `BSK1E` $= e^x K_1(x) \approx 1/x$ overflows if x is too close to zero.

BSJNS

Evaluates a sequence of Bessel functions of the first kind with integer order and real or complex arguments.

Required Arguments

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)
The absolute value of real arguments must be less than 10^5.
The absolute value of complex arguments must be less than 10^4.

N — Number of elements in the sequence. (Input)
It must be a positive integer.

BS — Vector of length `N` containing the values of the function through the series. (Output)
`BS(I)` contains the value of the Bessel function of order `I` − 1 at x for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL BSJNS (X, N, BS)`

Specific: The specific interface names are `S_BSJNS`, `D_BSJNS`, `C_BSJNS`, and `Z_BSJNS`

FORTRAN 77 Interface

Single: `CALL BSJNS (X, N, BS)`

Double: The double precision name is `DBSJNS`.

Complex: The complex name is `CBJNS`.

Double Complex: The double complex name is `DCBJNS`.

Example

In this example, $J_n(10.0)$, $n = 0, \ldots, 9$ is computed and printed.

```
      USE BSJNS_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    N
      PARAMETER  (N=10)
!
      INTEGER    K, NOUT
      REAL       BS(N), X
 !                                  Compute
      X = 10.0
      CALL BSJNS (X, N, BS)
!                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, BS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', I2, ' (', F6.3, ') = ', F6.3)
      END
```

Output

```
J sub  0 (10.000) = -0.246
J sub  1 (10.000) =  0.043
J sub  2 (10.000) =  0.255
J sub  3 (10.000) =  0.058
J sub  4 (10.000) = -0.220
J sub  5 (10.000) = -0.234
J sub  6 (10.000) = -0.014
J sub  7 (10.000) =  0.217
J sub  8 (10.000) =  0.318
J sub  9 (10.000) =  0.292
```

Description

The complex Bessel function $J_n(z)$ is defined to be

$$J_n(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - n\theta)\,d\theta$$

This code is based on the work of Sookne (1973a) and Olver and Sookne (1972). It uses backward recursion with strict error control.

Additional Example

In this example, $J_n(10 + 10i)$, $n = 0, \ldots, 10$ is computed and printed.

```
      USE BSJNS_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    N
      PARAMETER  (N=11)
!
      INTEGER    K, NOUT
      COMPLEX    CBS(N), Z
!                                  Compute
      Z = (10.0, 10.0)
      CALL BSJNS (Z, N, CBS)
!                                   Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', I2, ' ((', F6.3, ',', F6.3, &
           ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
J sub  0 ((10.000,10.000)) = (-2314.975,  411.563)
J sub  1 ((10.000,10.000)) = ( -460.681,-2246.627)
J sub  2 ((10.000,10.000)) = ( 2044.245, -590.157)
J sub  3 ((10.000,10.000)) = (  751.498, 1719.746)
J sub  4 ((10.000,10.000)) = (-1302.871,  880.632)
J sub  5 ((10.000,10.000)) = ( -920.394, -846.345)
J sub  6 ((10.000,10.000)) = (  419.501, -843.607)
J sub  7 ((10.000,10.000)) = (  665.930,   88.480)
J sub  8 ((10.000,10.000)) = (  108.586,  439.392)
J sub  9 ((10.000,10.000)) = ( -227.548,  176.165)
J sub 10 ((10.000,10.000)) = ( -154.831,  -76.050)
```

BSINS

Evaluates a sequence of modified Bessel functions of the first kind with integer order and real or complex arguments.

Required Arguments

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)
For real argument exp(|x|) must not overflow. For complex arguments x must be less than 10^4 in absolute value.

N — Number of elements in the sequence. (Input)

BSI — Vector of length `N` containing the values of the function through the series. (Output)
`BSI(I)` contains the value of the Bessel function of order `I` – 1 at *x* for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL BSINS (X, N, BSI)`

Specific: The specific interface names are `S_BSINS`, `D_BSINS`, `C_BSINS`, and `Z_BSINS`.

FORTRAN 77 Interface

Single: `CALL BSINS (X, N, BSI)`

Double: The double precision name is `DBSINS`.

Complex: The complex name is `CBINS`.

Double Complex: The double complex name is `DCBINS`.

Example

In this example, $I_n(10.0)$, $n = 0, \ldots, 10$ is computed and printed.

```
      USE BSINS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=11)
!
      INTEGER    K, NOUT
      REAL       BSI(N), X
!                                 Compute
      X = 10.0
      CALL BSINS (X, N, BSI)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, BSI(K)
   10 CONTINUE
99999 FORMAT (' I sub ', I2, ' (', F6.3, ') = ', F9.3)
      END
```

Output

```
I sub  0 (10.000) =  2815.716
I sub  1 (10.000) =  2670.988
I sub  2 (10.000) =  2281.519
I sub  3 (10.000) =  1758.381
I sub  4 (10.000) =  1226.490
I sub  5 (10.000) =   777.188
I sub  6 (10.000) =   449.302
I sub  7 (10.000) =   238.026
I sub  8 (10.000) =   116.066
I sub  9 (10.000) =    52.319
I sub 10 (10.000) =    21.892
```

Description

The complex Bessel function $I_n(z)$ is defined to be

$$I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(n\theta) \, d\theta$$

This code is based on the work of Sookne (1973a) and Olver and Sookne (1972). It uses backward recursion with strict error control.

Additional Example

In this example, $I_n(10 + 10i)$, $n = 0, \ldots, 10$ is computed and printed.

```
      USE BSINS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=11)
!
      INTEGER    K, NOUT
      COMPLEX    CBS(N), Z
!                                 Compute
      Z = (10.0, 10.0)
      CALL BSINS (Z, N, CBS)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' I sub ', I2, ' ((', F6.3, ',', F6.3, &
            ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
I sub  0 ((10.000,10.000)) = (-2314.975, -411.563)
I sub  1 ((10.000,10.000)) = (-2246.627, -460.681)
I sub  2 ((10.000,10.000)) = (-2044.245, -590.157)
I sub  3 ((10.000,10.000)) = (-1719.746, -751.498)
I sub  4 ((10.000,10.000)) = (-1302.871, -880.632)
I sub  5 ((10.000,10.000)) = ( -846.345, -920.394)
I sub  6 ((10.000,10.000)) = ( -419.501, -843.607)
I sub  7 ((10.000,10.000)) = (  -88.480, -665.930)
I sub  8 ((10.000,10.000)) = (  108.586, -439.392)
I sub  9 ((10.000,10.000)) = (  176.165, -227.548)
I sub 10 ((10.000,10.000)) = (  154.831,  -76.050)
```

BSJS

Evaluates a sequence of Bessel functions of the first kind with real order and real positive arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be at least zero and less than one.

X — Real argument for which the sequence of Bessel functions is to be evaluated. (Input)
It must be nonnegative.

N — Number of elements in the sequence. (Input)

BS — Vector of length `N` containing the values of the function through the series. (Output)
`BS(I)` contains the value of the Bessel function of order `XNU` + `I` – 1 at *x* for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL BSJS (XNU, X, N, BS)`

Specific: The specific interface names are `S_BSJS` and `D_BSJS`.

FORTRAN 77 Interface

Single: `CALL BSJS (XNU, X, N, BS)`

Double: The double precision name is `DBSJS`.

Example

In this example, $J_\nu(2.4048256)$, $\nu = 0, \ldots, 10$ is computed and printed.

```
      USE BSJS_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    N
      PARAMETER  (N=11)
!
      INTEGER    K, NOUT
      REAL       BS(N), X, XNU
!                                  Compute
      XNU = 0.0
      X   = 2.4048256
      CALL BSJS (XNU, X, N, BS)
!                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, X, BS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', F6.3, ' (', F6.3, ') = ', F10.3)
      END
```

Output

```
J sub  0.000 ( 2.405) =      0.000
J sub  1.000 ( 2.405) =      0.519
J sub  2.000 ( 2.405) =      0.432
J sub  3.000 ( 2.405) =      0.199
J sub  4.000 ( 2.405) =      0.065
J sub  5.000 ( 2.405) =      0.016
J sub  6.000 ( 2.405) =      0.003
J sub  7.000 ( 2.405) =      0.001
J sub  8.000 ( 2.405) =      0.000
J sub  9.000 ( 2.405) =      0.000
J sub 10.000 ( 2.405) =      0.000
```

Comments

Workspace may be explicitly provided, if desired, by use of `B2JS/DB2JS`. The reference is

```
CALL B2JS (XNU, X, N, BS, WK)
```

The additional argument is

WK — work array of length 2 ∗ `N`.

Description

The Bessel function $J_\nu(x)$ is defined to be

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^\pi \cos\left(x\cos\theta\right)\sin^{2\nu}\theta\, d\theta$$

This code is based on the work of Gautschi (1964) and Skovgaard (1975). It uses backward recursion.

BSYS

Evaluates a sequence of Bessel functions of the second kind with real nonnegative order and real positive arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be at least zero and less than one.

X — Real positive argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

BSY — Vector of length `N` containing the values of the function through the series. (Output)
`BSY(I)` contains the value of the Bessel function of order `I` – 1 + `XNU` at *x* for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL BSYS (XNU, X, N, BSY)`

Specific: The specific interface names are `S_BSYS` and `D_BSYS`.

FORTRAN 77 Interface

Single: `CALL BSYS (XNU, X, N, BSY)`

Double: The double precision name is DBSYS.

Example

In this example, $Y_{0.015625 + \nu - 1}(0.0078125)$, $\nu = 1, 2, 3$ is computed and printed.

```
      USE BSYS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=3)
!
      INTEGER    K, NOUT
      REAL       BSY(N), X, XNU
!                                 Compute
      XNU = 0.015625
      X   = 0.0078125
      CALL BSYS (XNU, X, N, BSY)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, X, BSY(K)
   10 CONTINUE
99999 FORMAT (' Y sub ', F6.3, ' (', F6.3, ') = ', F10.3)
      END
```

Output

```
Y sub  0.016 ( 0.008) =     -3.189
Y sub  1.016 ( 0.008) =    -88.096
Y sub  2.016 ( 0.008) = -22901.732
```

Description

The Bessel function $Y_\nu(x)$ is defined to be

$$Y_\nu(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta - \nu\theta)d\theta$$
$$-\frac{1}{\pi}\int_0^\infty \left[e^{\nu t} + e^{-\nu t}\cos(\nu\pi)\right]e^{-x\sinh t}\,dt$$

The variable ν must satisfy $0 \le \nu < 1$. If this condition is not met, then BS_i is set to $-b$. In addition, x must be in $[x_m, x_M]$ where $x_m = 6(16^{-32})$ and $x_M = 16^9$. If $x < x_m$, then $-b$ (b = AMACH(2), the largest representable number) is returned; and if $x > x_M$, then zero is returned.

The algorithm is based on work of Cody and others, (see Cody et al. 1976; Cody 1969; *NATS FUNPACK* 1976). It uses a special series expansion for small arguments. For moderate arguments, an analytic continuation in the argument based on Taylor series with special rational minimax approximations providing starting values is employed. An asymptotic expansion is used for large arguments.

BSIS

Evaluates a sequence of modified Bessel functions of the first kind with real order and real positive arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be greater than or equal to zero and less than one.

X — Real argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

BSI — Vector of length `N` containing the values of the function through the series. (Output)
`BSI(I)` contains the value of the Bessel function of order `I` – 1 + `XNU` at *x* for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL BSIS (XNU, X, N, BSI)`

Specific: The specific interface names are `S_BSIS` and `D_BSIS`.

FORTRAN 77 Interface

Single: `CALL BSIS (XNU, X, N, BSI)`

Double: The double precision name is `DBSIS`.

Example

In this example, $I_{\nu-1}(10.0)$, $\nu = 1, \ldots, 10$ is computed and printed.

```
      USE BSIS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
!
      INTEGER    K, NOUT
      REAL       BSI(N), X, XNU
!                                 Compute
      XNU = 0.0
      X   = 10.0
      CALL BSIS (XNU, X, N, BSI)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, X, BSI(K)
```

```
   10 CONTINUE
99999 FORMAT (' I sub ', F6.3, ' (', F6.3, ') = ', F10.3)
      END
```

Output

```
I sub  0.000 (10.000) =   2815.717
I sub  1.000 (10.000) =   2670.988
I sub  2.000 (10.000) =   2281.519
I sub  3.000 (10.000) =   1758.381
I sub  4.000 (10.000) =   1226.491
I sub  5.000 (10.000) =    777.188
I sub  6.000 (10.000) =    449.302
I sub  7.000 (10.000) =    238.026
I sub  8.000 (10.000) =    116.066
I sub  9.000 (10.000) =     52.319
```

Description

The Bessel function $I_\nu(x)$ is defined to be

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-x\cosh t - \nu t}\,dt$$

The input x must be nonnegative and less than or equal to log(b) (b = `AMACH`(2), the largest representable number). The argument ν = `XNU` must satisfy $0 \le \nu \le 1$.

Function `BSIS` is based on a code due to Cody (1983), which uses backward recursion.

BSIES

Evaluates a sequence of exponentially scaled modified Bessel functions of the first kind with nonnegative real order and real positive arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
It must be at least zero and less than one.

X — Real positive argument for which the sequence of Bessel functions is to be evaluated. (Input)
It must be nonnegative.

N — Number of elements in the sequence. (Input)

BSI — Vector of length `N` containing the values of the function through the series. (Output)
`BSI(I)` contains the value of the Bessel function of order `I` − 1 + `XNU` at x for `I` = 1 to `N` multiplied by exp(−`X`).

FORTRAN 90 Interface

Generic: `CALL BSIES (XNU, X, N, BSI)`

Specific: The specific interface names are `S_BSIES` and `D_BSIES`.

FORTRAN 77 Interface

Single: `CALL BSIES (XNU, X, N, BSI)`

Double: The double precision name is `DBSIES`.

Example

In this example, $I_{\nu-1}(10.0)$, $\nu = 1, \ldots, 10$ is computed and printed.

```
      USE BSIES_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
!
      INTEGER    K, NOUT
      REAL       BSI(N), X, XNU
!                                 Compute
      XNU = 0.0
      X   = 10.0
      CALL BSIES (XNU, X, N, BSI)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) X, XNU+K-1, X, BSI(K)
   10 CONTINUE
99999 FORMAT (' exp(-', F6.3, ') * I sub ', F6.3, &
           ' (', F6.3, ') = ', F6.3)
      END
```

Output

```
exp(-10.000) * I sub  0.000 (10.000) =  0.128
exp(-10.000) * I sub  1.000 (10.000) =  0.121
exp(-10.000) * I sub  2.000 (10.000) =  0.104
exp(-10.000) * I sub  3.000 (10.000) =  0.080
exp(-10.000) * I sub  4.000 (10.000) =  0.056
exp(-10.000) * I sub  5.000 (10.000) =  0.035
exp(-10.000) * I sub  6.000 (10.000) =  0.020
exp(-10.000) * I sub  7.000 (10.000) =  0.011
exp(-10.000) * I sub  8.000 (10.000) =  0.005
exp(-10.000) * I sub  9.000 (10.000) =  0.002
```

Description

Function BSIES evaluates $e^{-x} I_{\nu+k-1}(x)$, for $k = 1, \ldots, n$. For the definition of $I_\nu(x)$, see BSIS (page 117). The algorithm is based on a code due to Cody (1983), which uses backward recursion.

BSKS

Evaluates a sequence of modified Bessel functions of the second kind of fractional order.

Required Arguments

XNU — Fractional order of the function. (Input)
XNU must be less than one in absolute value.

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)

NIN — Number of elements in the sequence. (Input)

BK — Vector of length NIN containing the values of the function through the series. (Output)

FORTRAN 90 Interface

Generic: CALL BSKS (XNU, X, NIN, BK)

Specific: The specific interface names are S_BSKS and D_BSKS.

FORTRAN 77 Interface

Single: CALL BSKS (XNU, X, NIN, BK)

Double: The double precision name is DBSKS.

Example

In this example, $K_{\nu-1}(10.0)$, $\nu = 1, \ldots, 10$ is computed and printed.

```
      USE BSKS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NIN
      PARAMETER  (NIN=10)
!
      INTEGER    K, NOUT
      REAL       BS(NIN), X, XNU
!                                 Compute
      XNU = 0.0
      X   = 10.0
      CALL BSKS (XNU, X, NIN, BS)
!                                    Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, NIN
         WRITE (NOUT,99999) XNU+K-1, X, BS(K)
   10 CONTINUE
99999 FORMAT (' K sub ', F6.3, ' (', F6.3, ') = ', E10.3)
      END
```

Output

```
K sub  0.000 (10.000) =  0.178E-04
K sub  1.000 (10.000) =  0.186E-04
K sub  2.000 (10.000) =  0.215E-04
K sub  3.000 (10.000) =  0.273E-04
K sub  4.000 (10.000) =  0.379E-04
K sub  5.000 (10.000) =  0.575E-04
K sub  6.000 (10.000) =  0.954E-04
K sub  7.000 (10.000) =  0.172E-03
K sub  8.000 (10.000) =  0.336E-03
K sub  9.000 (10.000) =  0.710E-03
```

Comments

1. If `NIN` is positive, `BK`(1) contains the value of the function of order `XNU`, `BK`(2) contains the value of the function of order `XNU` + 1, … and `BK`(`NIN`) contains the value of the function of order `XNU` + `NIN` – 1.

2. If `NIN` is negative, `BK`(1) contains the value of the function of order `XNU`, `BK`(2) contains the value of the function of order `XNU` – 1, … and `BK`(`ABS`(`NIN`)) contains the value of the function of order `XNU` + `NIN` + 1.

Description

The Bessel function $K_\nu(x)$ is defined to be

$$K_\nu(x) = \frac{\pi}{2} e^{\nu\pi i/2} \left[i J_\nu(xe^{\frac{\pi}{2}i}) - Y_\nu(xe^{\frac{\pi}{2}i}) \right] \quad \text{for } -\pi < \arg x \le \frac{\pi}{2}$$

Currently, ν is restricted to be less than one in absolute value. A total of $|n|$ values is stored in the array `BK`. For positive n, `BK`(1) = $K_\nu(x)$, `BK`(2) = $K_{\nu+1}(x)$, …, `BK`(n) = $K_{\nu+n-1}(x)$. For negative n, `BK`(1) = $K_\nu(x)$, `BK`(2) = $K_{\nu-1}(x)$, …, `BK`($|n|$) = $K_{\nu+n+1}$.

`BSKS` is based on the work of Cody (1983).

BSKES

Evaluates a sequence of exponentially scaled modified Bessel functions of the second kind of fractional order.

Required Arguments

XNU — Fractional order of the function. (Input)
`XNU` must be less than 1.0 in absolute value.

X — Argument for which the sequence of Bessel functions is to be evaluated. (Input)

NIN — Number of elements in the sequence. (Input)

BKE — Vector of length `NIN` containing the values of the function through the series. (Output)

FORTRAN 90 Interface

Generic: `CALL BSKES (XNU, X, NIN, BKE)`

Specific: The specific interface names are `S_BSKES` and `D_BSKES`.

FORTRAN 77 Interface

Single: `CALL BSKES (XNU, X, NIN, BKE)`

Double: The double precision name is `DBSKES`.

Example

In this example, $K_{\nu-1/2}(2.0)$, $\nu = 1, \ldots, 6$ is computed and printed.

```
      USE BSKES_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NIN
      PARAMETER  (NIN=6)
!
      INTEGER    K, NOUT
      REAL       BKE(NIN), X, XNU
!                                  Compute
      XNU = 0.5
      X   = 2.0
      CALL BSKES (XNU, X, NIN, BKE)
!                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, NIN
         WRITE (NOUT,99999) X, XNU+K-1, X, BKE(K)
   10 CONTINUE
99999 FORMAT (' exp(', F6.3, ') * K sub ', F6.3, &
           ' (', F6.3, ') = ', F8.3)
      END
```

Output

```
exp( 2.000) * K sub  0.500 ( 2.000) =    0.886
exp( 2.000) * K sub  1.500 ( 2.000) =    1.329
exp( 2.000) * K sub  2.500 ( 2.000) =    2.880
exp( 2.000) * K sub  3.500 ( 2.000) =    8.530
exp( 2.000) * K sub  4.500 ( 2.000) =   32.735
exp( 2.000) * K sub  5.500 ( 2.000) =  155.837
```

Comments

1. If `NIN` is positive, `BKE`(1) contains `EXP`(`X`) times the value of the function of order `XNU`, `BKE`(2) contains `EXP`(`X`) times the value of the function of order `XNU` + 1, …, and `BKE`(`NIN`) contains `EXP`(`X`) times the value of the function of order `XNU` + `NIN` − 1.

2. If `NIN` is negative, `BKE`(1) contains `EXP`(`X`) times the value of the function of order `XNU`, `BKE`(2) contains `EXP`(`X`) times the value of the function of order `XNU` − 1, …, and `BKE`(`ABS`(`NIN`)) contains `EXP`(`X`) times the value of the function of order `XNU` + `NIN` + 1.

Description

Function `BSKES` evaluates $e^x K_{\nu+k-1}(x)$, for $k = 1, \ldots, n$. For the definition of $K_\nu(x)$, see `BSKS` (page 120).

Currently, ν is restricted to be less than 1 in absolute value. A total of $|n|$ values is stored in the array `BKE`. For n positive, `BKE`(1) contains $e^x K\nu(x)$, `BKE`(2) contains $e^x K_{\nu+1}(x)$, …, and `BKE`(`N`) contains $e^x K_{\nu+n-1}(x)$. For n negative, `BKE`(1) contains $e^x K_\nu(x)$, `BKE`(2) contains $e^x K_{\nu-1}(x)$, …, and `BKE`($|n|$) contains $e^x K_{\nu+n+1}(x)$. This routine is particularly useful for calculating sequences for large x provided $n \le x$. (Overflow becomes a problem if $n << x$.) n must not be zero, and x must not be greater than zero. Moreover, $|\nu|$ must be less than 1. Also, when $|n|$ is large compared with x, $|\nu + n|$ must not be so large that $e^x K_{\nu+n}(x) \approx e^x\Gamma(|\nu + n|)/[2(x/2)^{|\nu + n|}]$ overflows.

`BSKES` is based on the work of Cody (1983).

CBJS

Evaluates a sequence of Bessel functions of the first kind with real order and complex arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` – 1 at `Z` for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL CBJS (XNU, Z, N, CBS)`

Specific: The specific interface names are `S_CBJS` and `D_CBJS`.

FORTRAN 77 Interface

Single: `CALL CBJS (XNU, Z, N, CBS)`

Double: The double precision name is `DCBJS`.

Example

In this example, $J_{0.3+\nu-1}(1.2 + 0.5i)$, ν = 1, …, 4 is computed and printed.

```
      USE CBJS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=4)
!
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
!                                 Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBJS (XNU, Z, N, CBS)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' J sub ', F6.3, ' ((', F6.3, ',', F6.3, &
           ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
J sub  0.300 (( 1.200, 0.500)) = (    0.774,   -0.107)
J sub  1.300 (( 1.200, 0.500)) = (    0.400,    0.159)
J sub  2.300 (( 1.200, 0.500)) = (    0.087,    0.092)
J sub  3.300 (( 1.200, 0.500)) = (    0.008,    0.024)
```

Comments

Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Description

The Bessel function $J_\nu(z)$ is defined to be

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\theta - \nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{z\sinh t - \nu t}dt$$

$$\text{for } |\arg z| < \frac{\pi}{2}$$

This code is based on the code `BESSCC` of Barnett (1981) and Thompson and Barnett (1987).

This code computes $J_\nu(z)$ from the modified Bessel function $I_\nu(z)$ (see page 129), using the following relation, with $\rho = e^{i\pi/2}$:

$$Y_\nu(z) = \begin{cases} \rho I_\nu(z/\rho) & \text{for} -\pi/2 < \arg z \le \pi \\ \rho^3 I_\nu(\rho^3 z) & \text{for} -\pi < \arg z \le \pi/2 \end{cases}$$

CBYS

Evaluates a sequence of Bessel functions of the second kind with real order and complex arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at `Z` for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL CBYS (XNU, Z, N, CBS)`

Specific: The specific interface names are `S_CBYS` and `D_CBYS`.

FORTRAN 77 Interface

Single: `CALL CBYS (XNU, Z, N, CBS)`

Double: The double precision name is `DCBYS`.

Example

In this example, $Y_\nu 0.3 + n - 1(1.2 + 0.5i)$, $\nu = 1, \ldots, 4$ is computed and printed.

```
      USE CBYS_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=4)
!
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
!                                 Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBYS (XNU, Z, N, CBS)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' Y sub ', F6.3, ' ((', F6.3, ',', F6.3, &
           ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
Y sub  0.300 (( 1.200, 0.500)) = (   -0.013,    0.380)
Y sub  1.300 (( 1.200, 0.500)) = (   -0.716,    0.338)
Y sub  2.300 (( 1.200, 0.500)) = (   -1.048,    0.795)
Y sub  3.300 (( 1.200, 0.500)) = (   -1.625,    3.684)
```

Comments

1. Workspace may be explicitly provided, if desired, by use of `C2YS/DC2Y`.

   The reference is:

   `CALL C2YS (XNU, Z, N, CBS, FK)`

   The additional argument is:

   FK — complex work vector of length N.

2. Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Description

The Bessel function $Y_\nu(z)$ is defined to be

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - \nu\theta)\,d\theta$$
$$-\frac{1}{\pi}\int_0^\infty \left[e^{\nu t} + e^{-\nu t}\cos(\nu\pi)\right]e^{z\sinh t}\,dt$$
$$\text{for } |\arg z| < \frac{\pi}{2}$$

This code is based on the code `BESSEC` of Barnett (1981) and Thompson and Barnett (1987).

This code computes $Y_\nu(z)$ from the modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ (see `CBIS`, page 127, and `CBKS`, page 129), using the following relation:

$$Y_\nu(ze^{\pi i/2}) = e^{(\nu+1)\pi i/2} I_\nu(z) - \frac{2}{\pi}e^{-\nu\pi i/2}K_\nu(z) \quad \text{for } -\pi < \arg z \le \pi/2$$

CBIS

Evaluates a sequence of modified Bessel functions of the first kind with real order and complex arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at `Z` for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL CBIS (XNU, Z, N, CBS)`

Specific: The specific interface names are `S_CBIS` and `D_CBIS`.

FORTRAN 77 Interface

Single: `CALL CBIS (XNU, Z, N, CBS)`

Double: The double precision name is `DCBIS`.

Example

In this example, $I_{0.3 + \nu - 1}(1.2 + 0.5i)$, $\nu = 1, \ldots, 4$ is computed and printed.

```
      USE CBIS_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    N
      PARAMETER  (N=4)
!
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
!                                  Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBIS (XNU, Z, N, CBS)
!                                  Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' I sub ', F6.3, ' ((', F6.3, ',', F6.3, &
              ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
I sub  0.300 (( 1.200, 0.500)) = (    1.163,    0.396)
I sub  1.300 (( 1.200, 0.500)) = (    0.447,    0.332)
I sub  2.300 (( 1.200, 0.500)) = (    0.082,    0.127)
I sub  3.300 (( 1.200, 0.500)) = (    0.006,    0.029)
```

Comments

Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in `CBS` are valid.

Description

The modified Bessel function $I_\nu(z)$ is defined to be

$$I_v(z) = e^{-v\pi i/2} J_v(ze^{\pi i/2}) \quad \text{for} - \pi < \arg z \le \frac{\pi}{2}$$

where the Bessel function $J_\nu(z)$ is defined in `BSJS` (page 113).

This code is based on the code `BESSCC` of Barnett (1981) and Thompson and Barnett (1987).

For large arguments, *z*, Temme's (1975) algorithm is used to find $I_\nu(z)$. The $I_\nu(z)$ values are recurred upward (if this is stable). This involves evaluating a continued fraction. If this evaluation fails to converge, the answer may not be accurate. For moderate and small arguments, Miller's method is used.

CBKS

Evaluates a sequence of modified Bessel functions of the second kind with real order and complex arguments.

Required Arguments

XNU — Real argument which is the lowest order desired. (Input)
`XNU` must be greater than −1/2.

Z — Complex argument for which the sequence of Bessel functions is to be evaluated. (Input)

N — Number of elements in the sequence. (Input)

CBS — Vector of length `N` containing the values of the function through the series. (Output)
`CBS(I)` contains the value of the Bessel function of order `XNU` + `I` − 1 at `Z` for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL CBKS (XNU, Z, N, CBS)`

Specific: The specific interface names are `S_CBKS` and `D_CBKS`.

FORTRAN 77 Interface

Single: `CALL CBKS (XNU, Z, N, CBS)`

Double: The double precision name is `DCBKS`.

Example

In this example, $K_{0.3+\nu-1}(1.2 + 0.5i)$, ν = 1, …, 4 is computed and printed.

```
      USE UMACH_INT
      USE CBKS_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=4)
!
      INTEGER    K, NOUT
      REAL       XNU
      COMPLEX    CBS(N), Z
!                                 Compute
      XNU = 0.3
      Z   = (1.2, 0.5)
      CALL CBKS (XNU, Z, N, CBS)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) XNU+K-1, Z, CBS(K)
   10 CONTINUE
99999 FORMAT (' K sub ', F6.3, ' ((', F6.3, ',', F6.3, &
          ')) = (', F9.3, ',', F9.3, ')')
      END
```

Output

```
K sub  0.300 (( 1.200, 0.500)) = (    0.246,   -0.200)
K sub  1.300 (( 1.200, 0.500)) = (    0.336,   -0.362)
K sub  2.300 (( 1.200, 0.500)) = (    0.587,   -1.126)
K sub  3.300 (( 1.200, 0.500)) = (    0.719,   -4.839)
```

Comments

Workspace may be explicitly provided, if desired, by use of C2KS/DC2KS. The reference is

```
CALL C2KS (XNU, Z, N, CBS, FK)
```

The additional argument is

FK — Complex work vector of length N.

Informational errors

Type	Code	
3	1	One of the continued fractions failed.
4	2	Only the first several entries in CBS are valid.

Description

The Bessel function $K_\nu(z)$ is defined to be

$$K_\nu(z) = \frac{\pi}{2} e^{\nu\pi i/2} \left[iJ_\nu(ze^{\pi i/2}) - Y_\nu(ze^{\pi i/2}) \right] \qquad \text{for} -\pi < \arg z \le \frac{\pi}{2}$$

where the Bessel function $J_\nu(z)$ is defined in CBJS (page 123) and $Y_\nu(z)$ is defined in CBYS (page 125).

This code is based on the code `BESSCC` of Barnett (1981) and Thompson and Barnett (1987).

For moderate or large arguments, z, Temme's (1975) algorithm is used to find $K_\nu(z)$. This involves evaluating a continued fraction. If this evaluation fails to converge, the answer may not be accurate. For small z, a Neumann series is used to compute $K_\nu(z)$. Upward recurrence of the $K_\nu(z)$ is always stable.

Chapter 7: Kelvin Functions

Routines

Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964). The Kelvin functions are related to the Bessel functions by the following relations.

$$\text{ber}_\nu x + i\,\text{bei}_\nu x = J_\nu(xe^{3\pi i/4})$$

$$\text{ker}_\nu x + i\,\text{kei}_\nu x = e^{-\nu\pi i/2} K_\nu(xe^{\pi i/4})$$

The derivatives of the Kelvin functions are related to the values of the Kelvin functions by the following:

$$\sqrt{2}\text{ber}'_0 x = \text{ber}_1 x + \text{bei}_1 x$$

$$\sqrt{2}\text{bei}'_0 x = -\text{ber}_1 x + \text{bei}_1 x$$

$$\sqrt{2}\text{ker}'_0 x = \text{ker}_1 x + \text{kei}_1 x$$

$$\sqrt{2}\text{kei}'_0 x = -\text{ker}_1 x + \text{kei}_1 x$$

Plots of $\mathrm{ber}_n(x)$, $\mathrm{bei}_n(x)$, $\mathrm{ker}_n(x)$ and $\mathrm{kei}_n(x)$ for $n = 0$, 1 follow:

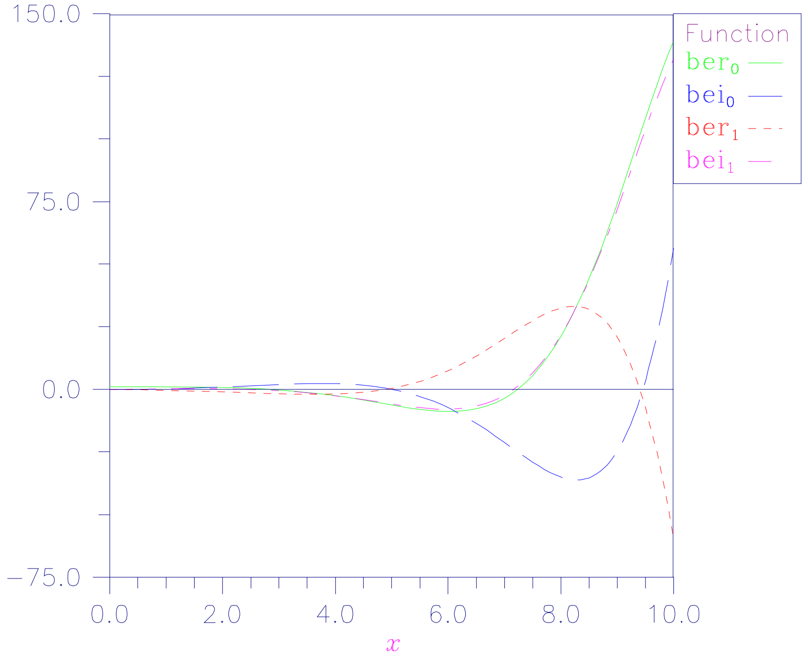


Figure 7-1 Plot of $\mathrm{ber}_n(x)$ and $\mathrm{bei}_n(x)$

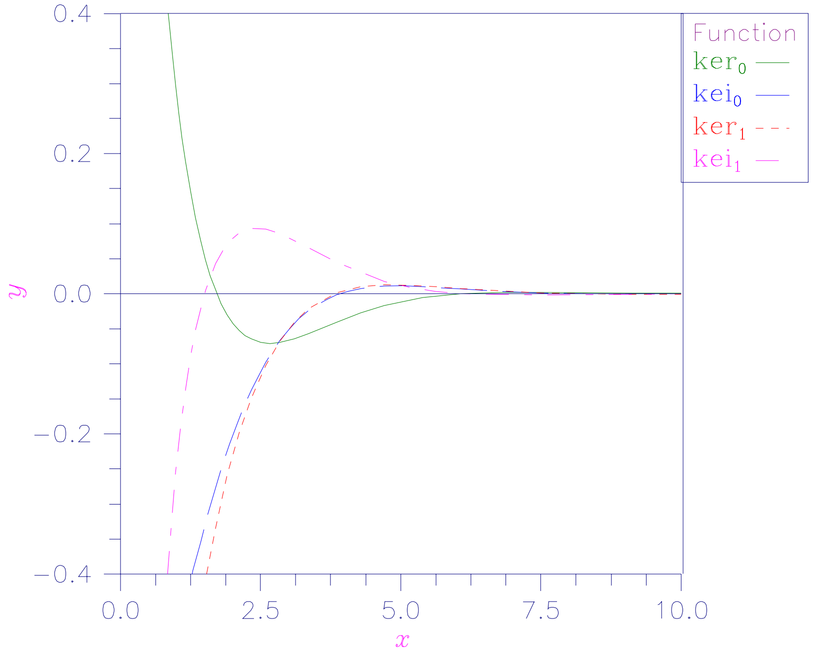


Figure 7-2 Plot of $\mathrm{ker}_n(x)$ and $\mathrm{kei}_n(x)$

BER0

This function evaluates the Kelvin function of the first kind, ber, of order zero.

Function Return Value

BER0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
`ABS(X)` must be less than 119.

FORTRAN 90 Interface

Generic: `BER0 (X)`

Specific: The specific interface names are `S_BER0` and `D_BER0`.

FORTRAN 77 Interface

Single: `BER0 (X)`

Double: The double precision name is `DBER0`.

Example

In this example, $\mathrm{ber}_0(0.4)$ is computed and printed.

```
      USE BER0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = BER0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BER0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BER0( 0.400) = 1.000
```

Description

The Kelvin function $\mathrm{ber}_0(x)$ is defined to be $\Re J_0(xe^{3\pi i/4})$. The Bessel function $J_0(x)$ is defined in BSJ0 (page 92). Function BER0 is based on the work of Burgoyne (1963).

BEI0

This function evaluates the Kelvin function of the first kind, bei, of order zero.

Function Return Value

BEI0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
ABS(X) must be less than 119.

FORTRAN 90 Interface

Generic: BEI0 (X)

Specific: The specific interface names are S_BEI0 and D_BEI0.

FORTRAN 77 Interface

Single: BEI0 (X)

Double: The double precision name is DBEI0.

Example

In this example, $\mathrm{bei}_0(0.4)$ is computed and printed.

```
      USE BEI0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = BEI0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BEI0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BEI0( 0.400) = 0.040
```

Description

The Kelvin function $\mathrm{bei}_0(x)$ is defined to be $\Im J_0(xe^{3\pi i/4})$. The Bessel function $J_0(x)$ is defined in `BSJ0` (page 92). Function `BEI0` is based on the work of Burgoyne (1963).

In `BEI0`, x must be less than 119.

AKER0

This function evaluates the Kelvin function of the second kind, ker, of order zero.

Function Return Value

AKER0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `AKER0(X)`

Specific: The specific interface names are `S_AKER0` and `D_AKER0`.

FORTRAN 77 Interface

Single: `AKER0(X)`

Double: The double precision name is `DKER0`.

Example

In this example, $\mathrm{ker}_0(0.4)$ is computed and printed.

```
      USE AKER0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = AKER0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKER0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKER0( 0.400) = 1.063
```

Description

The modified Kelvin function $\ker_0(x)$ is defined to be $\Re K_0(xe^{\pi i/4})$. The Bessel function $K_0(x)$ is defined in `BSK0` (page 101). Function `AKER0` is based on the work of Burgoyne (1963). If $x < 0$, then NaN (not a number) is returned. If $x \geq 119$, then zero is returned.

AKEI0

This function evaluates the Kelvin function of the second kind, kei, of order zero.

Function Return Value

AKEI0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative and less than 119.

FORTRAN 90 Interface

Generic: `AKEI0(X)`

Specific: The specific interface names are `S_AKEI0` and `D_AKEI0`.

FORTRAN 77 Interface

Single: `AKEI0(X)`

Double: The double precision name is `DKEI0`.

Example

In this example, $\text{kei}_0(0.4)$ is computed and printed.

```
      USE AKEI0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = AKEI0(X)
!                                   Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKEI0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKEI0( 0.400) = -0.704
```

Description

The modified Kelvin function $\text{kei}_0(x)$ is defined to be $\Im K_0(xe^{\pi i/4})$. The Bessel function $K_0(x)$ is defined in `BSK0` (page 101). Function `AKEI0` is based on the work of Burgoyne (1963).

In `AKEI0`, x must satisfy $0 \le x < 119$. If $x < 0$, then NaN (not a number) is returned. If $x \ge 119$, then zero is returned.

BERP0

This function evaluates the derivative of the Kelvin function of the first kind, ber, of order zero.

Function Return Value

BERP0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BERP0 (X)`

Specific: The specific interface names are `S_BERP0` and `D_BERP0`.

FORTRAN 77 Interface

Single: `BERP0 (X)`

Double: The double precision name is `DBERP0`.

Example

In this example, $\text{ber}'_0(0.6)$ is computed and printed.

```
      USE BERP0_INT
      USE UMACH_INT
```

```
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.6
      VALUE = BERP0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BERP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BERP0( 0.600) = -0.013
```

Description

The function $\mathrm{ber}'_0(x)$ is defined to be

$$\frac{d}{dx}\mathrm{ber}_0(x)$$

where $\mathrm{ber}_0(x)$ is a Kelvin function, see `BER0` (page 135). Function `BERP0` is based on the work of Burgoyne (1963).

If |x| > 119, then NaN (not a number) is returned.

BEIP0

This function evaluates the derivative of the Kelvin function of the first kind, bei, of order zero.

Function Return Value

BEIP0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BEIP0 (X)`

Specific: The specific interface names are `S_BEIP0` and `D_BEIP0`.

FORTRAN 77 Interface

Single: `BEIP0 (X)`

Double: The double precision name is `DBEIP0`.

Example

In this example, $\mathrm{bei}'_0(0.6)$ is computed and printed.

```
      USE BEIP0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.6
      VALUE = BEIP0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BEIP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
BEIP0( 0.600) = 0.300
```

Description

The function $\mathrm{bei}'_0(x)$ is defined to be

$$\frac{d}{dx}\,\mathrm{bei}_0(x)$$

where $\mathrm{bei}_0(x)$ is a Kelvin function, see `BEI0` (page 136). Function `BEIP0` is based on the work of Burgoyne (1963).

If $|x| > 119$, then NaN (not a number) is returned.

AKERP0

This function evaluates the derivative of the Kelvin function of the second kind, ker, of order zero.

Function Return Value

AKERP0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `AKERP0 (X)`

Specific: The specific interface names are `S_AKERP0` and `D_AKERP0`.

FORTRAN 77 Interface

Single: `AKERP0` (X)

Double: The double precision name is `DKERP0`.

Example

In this example, $\mathrm{ker}'_0(0.6)$ is computed and printed.

```
      USE AKERP0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.6
      VALUE = AKERP0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKERP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKERP0( 0.600) = -1.457
```

Description

The function $\mathrm{ker}'_0(x)$ is defined to be

$$\frac{d}{dx}\,\mathrm{ker}_0(x)$$

where $\mathrm{ker}_0(x)$ is a Kelvin function, see `AKER0` (page 137). Function `AKERP0` is based on the work of Burgoyne (1963). If $x < 0$, then NaN (not a number) is returned. If $x > 119$, then zero is returned.

AKEIP0

This function evaluates the Kelvin function of the second kind, kei, of order zero.

Function Return Value

AKEIP0 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `AKEIP0 (X)`

Specific: The specific interface names are `S_AKEIP0` and `D_AKEIP0`.

FORTRAN 77 Interface

Single: `AKEIP0 (X)`

Double: The double precision name is `DKEIP0`.

Example

In this example, $\mathrm{kei}'_0(0.6)$ is computed and printed.

```
      USE AKEIP0_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.6
      VALUE = AKEIP0(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKEIP0(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKEIP0( 0.600) = 0.348
```

Description

The function $\mathrm{kei}'_0(x)$ is defined to be

$$\frac{d}{dx}\,\mathrm{kei}_0(x)$$

where $\mathrm{kei}_0(x)$ is a Kelvin function, see `AKEIP0` (page 142). Function `AKEIP0` is based on the work of Burgoyne (1963).

If $x < 0$, then NaN (not a number) is returned. If $x > 119$, then zero is returned.

BER1

This function evaluates the Kelvin function of the first kind, ber, of order one.

Function Return Value

BER1 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BER1 (X)`

Specific: The specific interface names are `S_BER1` and `D_BER1`.

FORTRAN 77 Interface

Single: `BER1 (X)`

Double: The double precision name is `DBER1`.

Example

In this example, $\mathrm{ber}_1(0.4)$ is computed and printed.

```
      USE BER1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = BER1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BER1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BER1( 0.400) = -0.144
```

Description

The Kelvin function $\mathrm{ber}_1(x)$ is defined to be $\Re J_1(xe^{3\pi i/4})$. The Bessel function $J_1(x)$ is defined in `BSJ1` (page 94). Function `BER1` is based on the work of Burgoyne (1963).

If $|x| > 119$, then NaN (not a number) is returned.

BEI1

This function evaluates the Kelvin function of the first kind, bei, of order one.

Function Return Value

BEI1 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BEI1 (X)`

Specific: The specific interface names are `S_BEI1` and `D_BEI1`.

FORTRAN 77 Interface

Single: `BEI1 (X)`

Double: The double precision name is `DBEI1`.

Example

In this example, $\mathrm{bei}_1(0.4)$ is computed and printed.

```
      USE BEI1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = BEI1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BEI1(', F6.3, ') = ', F6.3)
      END
```

Output

```
BEI1( 0.400) = 0.139
```

Description

The Kelvin function $\mathrm{bei}_1(x)$ is defined to be $\Im J_1(xe^{3\pi i/4})$. The Bessel function $J_1(x)$ is defined in `BSJ1` (page 94). Function `BEI1` is based on the work of Burgoyne (1963).

If $|x| > 119$, then NaN (not a number) is returned.

AKER1

This function evaluates the Kelvin function of the second kind, ker, of order one.

Function Return Value

AKER1 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `AKER1 (X)`

Specific: The specific interface names are `S_AKER1` and `D_AKER1`.

FORTRAN 77 Interface

Single: `AKER1 (X)`

Double: The double precision name is `DKER1`.

Example

In this example, $\mathrm{ker}_1(0.4)$ is computed and printed.

```
      USE AKER1_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = AKER1(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKER1(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKER1( 0.400) = -1.882
```

Description

The modified Kelvin function $\ker_1(x)$ is defined to be $e^{-\pi i/2}\Re K_1(xe^{\pi i/4})$. The Bessel function $K_1(x)$ is defined in BSK1 (page 103). Function AKER1 is based on the work of Burgoyne (1963).

If $x < 0$, then NaN (not a number) is returned. If $x \ge 119$, then zero is returned.

AKEI1

This function evaluates the Kelvin function of the second kind, kei, of order one.

Function Return Value

AKEI1 — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: AKEI1 (X)

Specific: The specific interface names are S_AKEI1 and D_AKEI1.

FORTRAN 77 Interface

Single: AKEI1 (X)

Double: The double precision name is DKEI1.

Example

In this example, $\mathrm{kei}_1(0.4)$ is computed and printed.

```
      USE UMACH_INT
      USE AKEI1_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.4
      VALUE = AKEI1(X)
!                                 Print the results
```

```
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AKEI1(', F6.3, ') = ', F6.3)
      END
```

Output

```
AKEI1( 0.400) = -1.444
```

Description

The modified Kelvin function $\mathrm{kei}_1(x)$ is defined to be $e^{-\pi i/2}\Im K_1(xe^{\pi i/4})$. The Bessel function $K_1(x)$ is defined in `BSK1` (page 103). Function `AKER1` is based on the work of Burgoyne (1963).

If $x < 0$, then NaN (not a number) is returned. If $x \geq 119$, then zero is returned.

Chapter 8: Airy Functions

Routines

AI

This function evaluates the Airy function.

Function Return Value

AI — Function value. (Output)

Required Arguments

X — Argument for which the Airy function is desired. (Input)

FORTRAN 90 Interface

Generic: `AI (X)`

Specific: The specific interface names are `S_AI` and `D_AI`.

FORTRAN 77 Interface

Single: `AI (X)`

Double: The double precision name is `DAI`.

Example

In this example, Ai(–4.9) is computed and printed.

```
      USE AI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = -4.9
      VALUE = AI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AI(', F6.3, ') = ', F6.3)
      END
```

Output

```
AI(-4.900) = 0.375
```

Comments

Informational error Type	Code	
2	1	The function underflows because X is greater than XMAX, where XMAX = (–3/2 ln(AMACH(1)))$^{2/3}$.

Description

The Airy function Ai(x) is defined to be

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{1}{3}t^3\right)dt = \sqrt{\frac{x}{3\pi^2}}K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

The Bessel function $K_\nu(x)$ is defined in BSKS (page 120).

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. Here, ε = AMACH(4) is the machine precision. Finally, x should be less than $x_{\max}$ so the answer does not underflow. Very approximately, $x_{\max} = \{-1.5 \ln s\}^{2/3}$, where s = AMACH(1), the smallest representable positive number. If underflows are a problem for large x, then the exponentially scaled routine AIE (page 154) should be used.

BI

This function evaluates the Airy function of the second kind.

Function Return Value

BI — Function value. (Output)

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BI (X)`

Specific: The specific interface names are `S_BI` and `D_BI`.

FORTRAN 77 Interface

Single: `BI (X)`

Double: The double precision name is `DBI`.

Example

In this example, Bi(−4.9) is computed and printed.

```
      USE BI_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = -4.9
      VALUE = BI(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BI(', F6.3, ') = ', F6.3)
      END
```

Output

```
BI(-4.900) = -0.058
```

Description

The Airy function of the second kind Bi(x) is defined to be

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt + \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt$$

It can also be expressed in terms of modified Bessel functions of the first kind, $I_\nu(x)$, and Bessel functions of the first kind, $J_\nu(x)$ (see `BSIS`, page 117, and `BSJS`, page 113):

$$\mathrm{Bi}(x) = \sqrt{\frac{x}{3}}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) + I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] \qquad \text{for } x > 0$$

and

$$\mathrm{Bi}(x)=\sqrt{-\frac{x}{3}}\left[J_{-1/3}\left(\frac{2}{3}|x|^{3/2}\right)-J_{1/3}\left(\frac{2}{3}|x|^{3/2}\right)\right] \quad \text{for } x<0$$

Let ε = `AMACH`(4), the machine precision. If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. In addition, x should not be so large that $\exp[(2/3)x^{3/2}]$ overflows. If overflows are a problem, consider using the exponentially scaled form of the Airy function of the second kind, `BIE` (page 155), instead.

AID

This function evaluates the derivative of the Airy function.

Function Return Value

AID — Function value. (Output)

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `AID (X)`

Specific: The specific interface names are `S_AID` and `D_AID`.

FORTRAN 77 Interface

Single: `AID (X)`

Double: The double precision name is `DAID`.

Example

In this example, Ai′(−4.9) is computed and printed.

```
      USE AID_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = -4.9
      VALUE = AID(X)
```

```
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AID(', F6.3, ') = ', F6.3)
      END
```

Output

```
AID(-4.900) = 0.147
```

Comments

Informational error

Type	Code	
2	1	The function underflows because X is greater than XMAX, where XMAX = –3/2 ln(AMACH(1)).

Description

The function Ai′(x) is defined to be the derivative of the Airy function, Ai(x) (see AI, page 149).

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. Here, ε = AMACH(4) is the machine precision. Finally, x should be less than $x_{\max}$ so that the answer does not underflow. Very approximately, $x_{\max} = \{-1.5 \ln s\}$, where s = AMACH(1), the smallest representable positive number. If underflows are a problem for large x, then the exponentially scaled routine AIDE (page 157) should be used.

BID

This function evaluates the derivative of the Airy function of the second kind.

Function Return Value

BID — Function value. (Output)

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BID (X)`

Specific: The specific interface names are `S_BID` and `D_BID`.

FORTRAN 77 Interface

Single: `BID (X)`

Double: The double precision name is DBID.

Example

In this example, $\mathrm{Bi}'(-4.9)$ is computed and printed.

```
      USE BID_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = -4.9
      VALUE = BID(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BID(', F6.3, ') = ', F6.3)
      END
```

Output

```
BID(-4.900) = 0.827
```

Description

The function $\mathrm{Bi}'(x)$ is defined to be the derivative of the Airy function of the second kind, $\mathrm{Bi}(x)$ (see BI, page 150).

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, the answer will be less accurate than half precision. In addition, x should not be so large that $\exp[(2/3)x^{3/2}]$ overflows. If overflows are a problem, consider using BIDE (page 158) instead. Here, ε = AMACH(4) is the machine precision.

AIE

This function evaluates the exponentially scaled Airy function.

Function Return Value

AIE — Function value. (Output)
The Airy function for negative arguments and the exponentially scaled Airy function, $e\,\zeta\mathrm{Ai}(\mathrm{X})$, for positive arguments where

$$\zeta = \tfrac{2}{3} X^{3/2}$$

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `AIE (X)`

Specific: The specific interface names are `S_AIE` and `D_AIE`.

FORTRAN 77 Interface

Single: `AIE (X)`

Double: The double precision name is `DAIE`.

Example

In this example, `AIE`(0.49) is computed and printed.

```
      USE AIE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.49
      VALUE = AIE(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AIE(', F6.3, ') = ', F6.3)
      END
```

Output

```
AIE( 0.490) = 0.294
```

Description

The exponentially scaled Airy function is defined to be

$$\mathrm{AIE}(x) = \begin{cases} \mathrm{Ai}(x) & \text{if } x \le 0 \\ e^{[2/3]x^{3/2}}\mathrm{Ai}(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

BIE

This function evaluates the exponentially scaled Airy function of the second kind.

Function Return Value

BIE — Function value. (Output)
The Airy function of the second kind for negative arguments and the exponentially scaled Airy function of the second kind, $e^{\zeta}\mathrm{Bi}(\mathtt{X})$, for positive arguments where

$$\zeta = -\tfrac{2}{3}\mathrm{X}^{3/2}$$

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BIE (X)`

Specific: The specific interface names are `S_BIE` and `D_BIE`.

FORTRAN 77 Interface

Single: `BIE (X)`

Double: The double precision name is `DBIE`.

Example

In this example, `BIE`(0.49) is computed and printed.

```
      USE BIE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.49
      VALUE = BIE(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BIE(', F6.3, ') = ', F6.3)
      END
```

Output

```
BIE( 0.490) = 0.675
```

Description

The exponentially scaled Airy function of the second kind is defined to be

$$\text{BIE}(x) = \begin{cases} \text{Bi}(x) & \text{if } x \le 0 \\ e^{-[2/3]x^{3/2}} \text{Bi}(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

AIDE

This function evaluates the exponentially scaled derivative of the Airy function.

Function Return Value

AIDE — Function value. (Output)
The derivative of the Airy function for negative arguments and the exponentially scaled derivative of the Airy function, $e^{\zeta}\text{Ai}'(\text{X})$, for positive arguments where

$$\zeta = -\tfrac{2}{3}\text{X}^{3/2}$$

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `AIDE (X)`

Specific: The specific interface names are `S_AIDE` and `D_AIDE`.

FORTRAN 77 Interface

Single: `AIDE (X)`

Double: The double precision name is `DAIDE`.

Example

In this example, `AIDE`(0.49) is computed and printed.

```
      USE AIDE_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                  Compute
      X     = 0.49
      VALUE = AIDE(X)
!                                  Print the results
```

```
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' AIDE(', F6.3, ') = ', F6.3)
      END
```

Output

```
AIDE( 0.490) = -0.284
```

Description

The exponentially scaled derivative of the Airy function is defined to be

$$\mathrm{AIDE}(x) = \begin{cases} \mathrm{Ai}'(x) & \text{if } x \le 0 \\ e^{[2/3]x^{3/2}}\mathrm{Ai}'(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = AMACH(4) is the machine precision.

BIDE

This function evaluates the exponentially scaled derivative of the Airy function of the second kind.

Function Return Value

BIDE — Function value. (Output)
The derivative of the Airy function of the second kind for negative arguments and the exponentially scaled derivative of the Airy function of the second kind, $e^{\zeta}\mathrm{Bi}'(\mathrm{X})$, for positive arguments where

$$\zeta = -\tfrac{2}{3}\mathrm{X}^{3/2}$$

Required Arguments

X — Argument for which the Airy function value is desired. (Input)

FORTRAN 90 Interface

Generic: `BIDE (X)`

Specific: The specific interface names are `S_BIDE` and `D_BIDE`.

FORTRAN 77 Interface

Single: `BIDE (X)`

Double: The double precision name is `DBIDE`.

Example

In this example, `BIDE`(0.49) is computed and printed.

```
      USE BIDE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.49
      VALUE = BIDE(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' BIDE(', F6.3, ') = ', F6.3)
      END
```

Output

```
BIDE( 0.490) = 0.430
```

Description

The exponentially scaled derivative of the Airy function of the second kind is defined to be

$$\mathtt{BIDE}(x) = \begin{cases} \mathrm{Bi}'(x) & \text{if } x \le 0 \\ e^{-[2/3]x^{3/2}}\mathrm{Bi}'(x) & \text{if } x > 0 \end{cases}$$

If $x < -1.31\varepsilon^{-2/3}$, then the answer will have no precision. If $x < -1.31\varepsilon^{-1/3}$, then the answer will be less accurate than half precision. Here, ε = `AMACH`(4) is the machine precision.

Chapter 9: Elliptic Integrals

Routines

Usage Notes

The notation used in this chapter follows that of Abramowitz and Stegun (1964) and Carlson (1979).

The complete elliptic integral of the first kind is

$$K(m) = \int_0^{\pi/2} \left(1 - m\sin^2\theta\right)^{-1/2} d\theta$$

and the complete elliptic integral of the second kind is

$$E(m) = \int_0^{\pi/2} \left(1 - m\sin^2\theta\right)^{1/2} d\theta$$

Instead of the *parameter* m, the *modular* angle α is sometimes used with $m = \sin^2\alpha$. Also used is the *modulus* k with $k^2 = m$.

$$\begin{aligned} E(k) &= \int_0^{\pi/2} \left(1 - k^2 \sin^2\theta\right)^{1/2} d\theta \\ &= R_F\left(0, 1-k^2, 1\right) - \frac{1}{3} k^2 R_D\left(0, 1-k^2, 1\right) \end{aligned}$$

Carlson Elliptic Integrals

The Carlson elliptic integrals are defined by Carlson (1979) as follows:

$$R_F(x, y, z) = \frac{1}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)\right]^{1/2}}$$

$$R_C(x, y) = \frac{1}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)^2\right]^{1/2}}$$

$$R_J(x, y, z, \rho) = \frac{3}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)(t+\rho)^2\right]^{1/2}}$$

$$R_D(x, y, z) = \frac{3}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)^3\right]^{1/2}}$$

The standard Legendre elliptic integrals can be written in terms of the Carlson functions as follows (these relations are from Carlson (1979)):

$$\begin{aligned} F(\phi, k) &= \int_0^\phi \left(1-k^2\sin^2\theta\right)^{-1/2} d\theta \\ &= (\sin\phi) R_F\left(\cos^2\phi, 1-k^2\sin^2\phi, 1\right) \end{aligned}$$

$$\begin{aligned} E(\phi, k) &= \int_0^\phi \left(1-k^2\sin^2\theta\right)^{1/2} d\theta \\ &= (\sin\phi) R_F\left(\cos^2\phi, 1-k^2\sin^2\phi, 1\right) - \frac{1}{3}k^2(\sin\phi)^3 R_D\left(\cos^2\phi, 1-k^2\sin^2\phi, 1\right) \end{aligned}$$

$$\begin{aligned} \Pi(\phi, k, n) &= \int_0^\phi \left(1+n\sin^2\theta\right)^{-1}\left(1-k^2\sin^2\theta\right)^{-1/2} d\theta \\ &= (\sin\phi) R_F\left(\cos^2\phi, 1-k^2\sin^2\phi, 1\right) - \frac{n}{3}k^2(\sin\phi)^3 R_D\left(\cos^2\phi, 1-k^2\sin^2\phi, 1+n\sin^2\phi\right) \end{aligned}$$

$$\begin{aligned} D(\phi, k, n) &= \int_0^\phi \sin^2\theta\left(1-k^2\sin^2\theta\right)^{-1/2} d\theta \\ &= \frac{1}{3}(\sin\phi)^3 R_D\left(\cos^2\phi, 1-k^2\sin^2\phi, 1\right) \end{aligned}$$

$$\begin{aligned} K(k) &= \int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{-1/2} d\theta \\ &= R_F\left(0, 1-k^2, 1\right) \end{aligned}$$

$$\begin{aligned} E(k) &= \int_0^{\pi/2}\left(1-k^2\sin^2\theta\right)^{1/2} d\theta \\ &= R_F\left(0, 1-k^2, 1\right) - \frac{1}{3}k^2 R_D\left(0, 1-k^2, 1\right) \end{aligned}$$

The function $R_C(x, y)$ is related to inverse trigonometric and inverse hyperbolic functions.

$$\ln x = (x-1)\, R_c\left[\left(\frac{1+x}{2}\right),\, x\right] \qquad 0 < x < \infty$$

$$\sin^{-1} x = xR_c\left(1-x^2, 1\right) \qquad -1 \le x \le 1$$

$$\sinh^{-1} x = xR_c\left(1+x^2, 1\right) \qquad -\infty < x < \infty$$

$$\cos^{-1} x = \sqrt{1-x^2}\, R_c\left(x^2, 1\right) \qquad 0 \le x \le 1$$

$$\cosh^{-1} x = \sqrt{x^2-1}\, R_c\left(x^2, 1\right) \qquad 1 \le x < \infty$$

$$\tan^{-1} x = xR_c\left(1, 1+x^2\right) \qquad -\infty < x < \infty$$

$$\tanh^{-1} x = xR_c\left(1, 1-x^2\right) \qquad -1 < x < 1$$

$$\cot^{-1} x = R_c\left(x^2,\, x^2+1\right) \qquad 0 < x < \infty$$

$$\coth^{-1} x = R_c\left(x^2, x^2-1\right) \qquad 1 < x < \infty$$

ELK

This function evaluates the complete elliptic integral of the kind `K`(x).

Function Return Value

ELK — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
`X` must be greater than or equal to 0 and less than 1.

FORTRAN 90 Interface

Generic: `ELK (X)`

Specific: The specific interface names are `S_ELK` and `D_ELK`.

FORTRAN 77 Interface

Single: `ELK (X)`

Double: The double precision name is `DELK`.

Example

In this example, $K(0)$ is computed and printed.

```
      USE ELK_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.0
      VALUE = ELK(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ELK(', F6.3, ') = ', F6.3)
      END
```

Output

```
ELK( 0.000) = 1.571
```

Description

The complete elliptic integral of the first kind is defined to be

$$K(x) = \int_0^{\pi/2} \frac{d\,\theta}{\left[1 - x\sin^2\theta\right]^{1/2}} \quad \text{for } 0 \le x < 1$$

The argument x must satisfy $0 \le x < 1$; otherwise, `ELK` is set to b = `AMACH`(2), the largest representable floating-point number.

The function $K(x)$ is computed using the routine `ELRF` (page 166) and the relation $K(x) = R_F(0, 1 - x, 1)$.

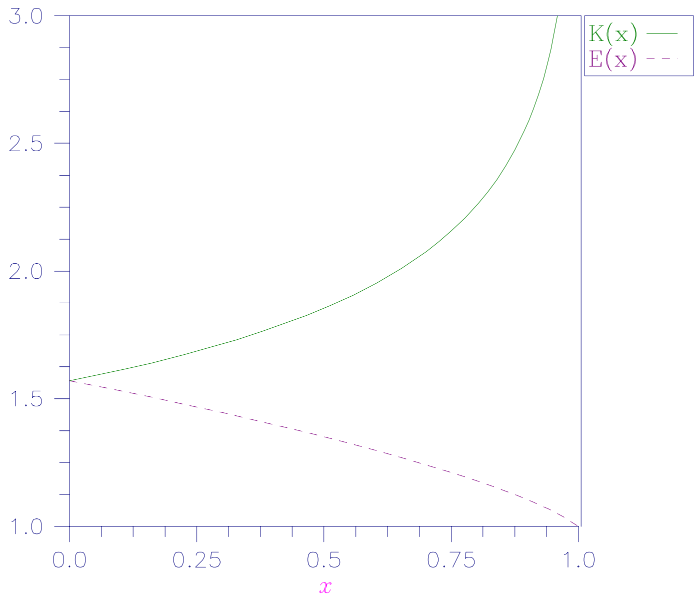


Figure 9-1 Plot of $K(x)$ and $E(x)$

ELE

This function evaluates the complete elliptic integral of the second kind $E(x)$.

Function Return Value

ELE — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)
`X` must be greater than or equal to 0 and less than or equal to 1.

FORTRAN 90 Interface

Generic: `ELE (X)`

Specific: The specific interface names are `S_ELE` and `D_ELE`.

FORTRAN 77 Interface

Single: `ELE (X)`

Double: The double precision name is `DELE`.

Example

In this example, $E(0.33)$ is computed and printed.

```
      USE ELE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.33
      VALUE = ELE(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' ELE(', F6.3, ') = ', F6.3)
      END
```

Output

```
ELE( 0.330) = 1.432
```

Description

The complete elliptic integral of the second kind is defined to be

$$E(x) = \int_0^{\pi/2} \left[1 - x\sin^2\theta\right]^{1/2} d\theta \quad \text{for } 0 \le x < 1$$

The argument x must satisfy $0 \le x < 1$; otherwise, `ELE` is set to b = `AMACH`(2), the largest representable floating-point number.

The function $E(x)$ is computed using the routines `ELRF`, page 166, and `ELRD`, page 167. The computation is done using the relation

$$E(x) = R_F(0, 1-x, 1) - \frac{x}{3} R_D(0, 1-x, 1)$$

For a plot of $E(x)$, see Figure 9.1 on page 164.

ELRF

This function evaluates Carlson's incomplete elliptic integral of the first kind R_F(`X`, `Y`, `Z`).

Function Return Value

ELRF — Function value. (Output)

Required Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative

Y — Second variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Z — Third variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

FORTRAN 90 Interface

Generic: `ELRF (X, Y, Z)`

Specific: The specific interface names are `S_ELRF` and `D_ELRF`.

FORTRAN 77 Interface

Single: `ELRF (X, Y, Z)`

Double: The double precision name is `DELRF`.

Example

In this example, $R_F(0, 1, 2)$ is computed and printed.

```
      USE ELRF_INT
      USE UMACH_INT
!                                  Declare variables
      INTEGER    NOUT
      REAL       VALUE, X, Y, Z
!                                    Compute
      X     = 0.0
      Y     = 1.0
      Z     = 2.0
      VALUE = ELRF(X, Y, Z)
!                                    Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, VALUE
99999 FORMAT (' ELRF(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ELRF( 0.000, 1.000, 2.000) = 1.311
```

Description

The Carlson's complete elliptic integral of the first kind is defined to be

$$R_F(x, y, z) = \frac{1}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)\right]^{1/2}}$$

The arguments must be nonnegative and less than or equal to $b/5$. In addition, $x + y$, $x + z$, and $y + z$ must be greater than or equal to $5s$. Should any of these conditions fail, ELRF is set to b. Here, b = AMACH(2) is the largest and s = AMACH(1) is the smallest representable floating-point number.

The function ELRF is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

ELRD

This function evaluates Carlson's incomplete elliptic integral of the second kind R_D(X, Y, Z).

Function Return Value

ELRD — Function value. (Output)

Required Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Y — Second variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Z — Third variable of the incomplete elliptic integral. (Input)
It must be positive.

FORTRAN 90 Interface

Generic: `ELRD(X, Y, Z)`

Specific: The specific interface names are `S_ELRD` and `D_ELRD`.

FORTRAN 77 Interface

Single: `ELRD(X, Y, Z)`

Double: The double precision name is `DELRD`.

Example

In this example, $R_D(0, 2, 1)$ is computed and printed.

```
      USE ELRD_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X, Y, Z
!                                 Compute
      X     = 0.0
      Y     = 2.0
      Z     = 1.0
      VALUE = ELRD(X, Y, Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, VALUE
99999 FORMAT (' ELRD(', F6.3, ',', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ELRD( 0.000, 2.000, 1.000) = 1.797
```

Description

The Carlson's complete elliptic integral of the second kind is defined to be

$$R_D(x, y, z) = \frac{3}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)(t+z)^3\right]^{1/2}}$$

The arguments must be nonnegative and less than or equal to $0.69(-\ln \varepsilon)^{1/9} s^{-2/3}$ where ε = `AMACH(4)` is the machine precision, s = `AMACH(1)` is the smallest representable positive number. Furthermore, $x + y$ and z must be greater than $\max\{3s^{2/3}, 3/b^{2/3}\}$, where b = `AMACH(2)` is the largest floating-point number. If any of these conditions are false, then `ELRD` is set to b.

The function `ELRD` is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

ELRJ

This function evaluates Carlson's incomplete elliptic integral of the third kind R_J(`X`, `Y`, `Z`, `RHO`)

Function Return Value

ELRJ — Function value. (Output)

Required Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Y — Second variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

Z — Third variable of the incomplete elliptic integral. (Input)
It must be nonnegative.

RHO — Fourth variable of the incomplete elliptic integral. (Input)
It must be positive.

FORTRAN 90 Interface

Generic: `ELRJ(X, Y, Z, RHO)`

Specific: The specific interface names are `S_ELRJ` and `D_ELRJ`.

FORTRAN 77 Interface

Single: `ELRJ(X, Y, Z, RHO)`

Double: The double precision name is `DELRJ`.

Example

In this example, $R_J(2, 3, 4, 5)$ is computed and printed.

```
      USE ELRJ_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       RHO, VALUE, X, Y, Z
!                                 Compute
      X     = 2.0
      Y     = 3.0
      Z     = 4.0
      RHO   = 5.0
      VALUE = ELRJ(X, Y, Z, RHO)
!                                  Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, Z, RHO, VALUE
99999 FORMAT (' ELRJ(', F6.3, ',', F6.3, ',', F6.3, ',', F6.3, &
           ') = ', F6.3)
      END
```

Output

```
ELRJ( 2.000, 3.000, 4.000, 5.000) = 0.143
```

Description

The Carlson's complete elliptic integral of the third kind is defined to be

$$R_J\left(x,\, y,\, z,\, \rho\right) = \frac{3}{2}\int_0^\infty \frac{dt}{\left[\left(t+x\right)\left(t+y\right)\left(t+z\right)\left(t+\rho\right)^2\right]^{1/2}}$$

The arguments must be nonnegative. In addition, $x + y$, $x + z$, $y + z$ and ρ must be greater than or equal to $(5s)^{1/3}$ and less than or equal to $.3(b/5)^{1/3}$, where s = `AMACH`(1) is the smallest representable floating-point number. Should any of these conditions fail, `ELRF` is set to b = `AMACH`(2), the largest floating-point number.

The function `ELRJ` is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

ELRC

This function evaluates an elementary integral from which inverse circular functions, logarithms and inverse hyperbolic functions can be computed.

Function Return Value

ELRC — Function value. (Output)

Required Arguments

X — First variable of the incomplete elliptic integral. (Input)
It must be nonnegative and satisfy the conditions given in Comments.

Y — Second variable of the incomplete elliptic integral. (Input)
It must be positive and satisfy the conditions given in Comments.

FORTRAN 90 Interface

Generic: `ELRC(X, Y)`

Specific: The specific interface names are S_ELRC and D_ELRC.

FORTRAN 77 Interface

Single: ELRC(X, Y)

Double: The double precision name is DELRC.

Example

In this example, $R_C(2.25, 2.0)$ is computed and printed.

```
      USE ELRC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X, Y
!                                 Compute
      X     = 0.0
      Y     = 1.0
      VALUE = ELRC(X, Y)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, Y, VALUE
99999 FORMAT (' ELRC(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ELRC( 0.000, 1.000) = 1.571
```

Comments

The sum X + Y must be greater than or equal to ARGMIN and both X and Y must be less than or equal to ARGMAX. ARGMIN = s * 5 and ARGMAX = $b/5$, where s is the machine minimum (AMACH(1)) and b is the machine maximum (AMACH(2)).

Description

The special case of Carlson's complete elliptic integral of the first kind is defined to be

$$R_C(x, y) = \frac{1}{2}\int_0^\infty \frac{dt}{\left[(t+x)(t+y)^2\right]^{1/2}}$$

The argument x must be nonnegative, y must be positive, and $x + y$ must be less than or equal to $b/5$ and greater than or equal to $5s$. If any of these conditions are false, then ELRC is set to b. Here, b = AMACH(2) is the largest and s = AMACH(1) is the smallest representable floating-point number.

The function ELRC is based on the code by Carlson and Notis (1981) and the work of Carlson (1979).

Chapter 10: Elliptic and Related Functions

Routines

Usage Notes

Elliptic functions are doubly periodic, single-valued complex functions of a single variable that are analytic, except at a finite number of poles. Because of the periodicity, we need consider only the fundamental period parallelogram. The irreducible number of poles, counting multiplicities, is the *order* of the elliptic function. The simplest, non-trivial, elliptic functions are of order two.

The Weierstrass elliptic functions, $\wp(z, \omega, \omega')$ have a double pole at $z = 0$ and so are of order two. Here, 2ω and $2\omega'$ are the periods.

The Jacobi elliptic functions each have two simple poles and so are also of order two. The period of the functions is as follows:

Function	**Periods**	
sn(x, m)	$4K(m)$	$2iK'(m)$
cn(x, m)	$4K(m)$	$4iK'(m)$
dn(x, m)	$2K(m)$	$4iK'(m)$

The function $K(m)$ is the complete elliptic integral, see `ELK` (page 163), and $K'(m) = K(1 - m)$.

CWPL

This function evaluates the Weierstrass' $\wp$ function in the lemniscatic case for complex argument with unit period parallelogram.

Function Return Value

CWPL — Complex function value. (Output)

Required Arguments

Z — Complex argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `CWPL (Z)`

Specific: The specific interface names are `C_CWPL` and `Z_CWPL`.

FORTRAN 77 Interface

Complex: `CWPL (Z)`

Double complex: The double complex name is `ZWPL`.

Example

In this example, $\wp(0.25 + 0.25i)$ is computed and printed.

```
      USE CWPL_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (0.25, 0.25)
      VALUE = CWPL(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPL(', F6.3, ',', F6.3, ') = (', &
            F6.3, ',', F6.3, ')')
      END
```

Output

```
CWPL( 0.250, 0.250) = ( 0.000,-6.875)
```

Description

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. `CWPL(Z)` computes $\wp(z \mid \omega, \omega')$ with $2\omega = 1$ and $2\omega' = i$.

The input argument is first reduced to the fundamental parallelogram of all z satisfying $-1/2 \le \Re z \le 1/2$ and $-1/2 \le \Im z \le 1/2$. Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points $z = m + ni$, which are the poles of `CWPL`. If the argument is a lattice point, then b = `AMACH`(2) , the largest floating-point number, is returned. If the argument has modulus greater than $10\varepsilon^{-1}$, then NaN (not a number) is returned. Here, ε = AMACH(4) is the machine precision.

Function CWPL is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

CWPLD

This function evaluates the first derivative of the Weierstrass' $\wp$ function in the lemniscatic case for complex argument with unit period parallelogram.

Function Return Value

CWPLD — Complex function value. (Output)

Required Arguments

Z — Complex argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: CWPLD (Z)

Specific: The specific interface names are C_CWPLD and Z_CWPLD.

FORTRAN 77 Interface

Complex: CWPLD (Z)

Double complex: The double complex name is ZWPLD.

Example

In this example, $\wp(0.25 + 0.25i)$ is computed and printed.

```
      USE CWPLD_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (0.25, 0.25)
      VALUE = CWPLD(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPLD(', F6.3, ',', F6.3, ') = (', &
           F6.3, ',', F6.3, ')')
      END
```

Output

```
CWPLD( 0.250, 0.250) = (36.054,36.054)
```

Description

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. CWPLD(Z) computes the derivative of $\wp(z \mid \omega, \omega')$ with $2\omega = 1$ and $2\omega' = i$. CWPL, page 173, computes $\wp(z \mid \omega, \omega')$.

The input argument is first reduced to the fundamental parallelogram of all z satisfying $-1/2 \le \Re z \le 1/2$ and $-1/2 \le \Im z \le 1/2$. Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points $z = m + ni$, which are the poles of CWPL. If the argument is a lattice point, then b = AMACH(2), the largest floating-point number, is returned.

Function CWPLD is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

CWPQ

This function evaluates the Weierstrass' $\wp$ function in the equianharmonic case for complex argument with unit period parallelogram.

Function Return Value

CWPQ — Complex function value. (Output)

Required Arguments

Z — Complex argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: CWPQ (Z)

Specific: The specific interface names are C_CWPQ and Z_CWPQ.

FORTRAN 77 Interface

Complex: CWPQ (Z)

Double complex: The double complex name is ZWPQ.

Example

In this example, $\wp(0.25 + 0.14437567i)$ is computed and printed.

```
      USE CWPQ_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (0.25, 0.14437567)
      VALUE = CWPQ(Z)
!                                 Print the results
```

```
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPQ(', F6.3, ',', F6.3, ') = (', &
            F7.3, ',', F7.3, ')')
      END
```

Output

```
CWPQ( 0.250, 0.144) = ( 5.895,-10.216)
```

Description

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. `CWPQ(Z)` computes $\wp(z \mid \omega, \omega')$ with

$$4\omega = 1 - i\sqrt{3} \text{ and } 4\omega' = 1 + i\sqrt{3}$$

The input argument is first reduced to the fundamental parallelogram of all z satisfying

$$-1/2 \le \Re z \le 1/2 \text{ and } -\sqrt{3}/4 \le \Im z \le \sqrt{3}/4$$

Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points

$$z = m\left(1 - i\sqrt{3}\right) + n\left(1 + i\sqrt{3}\right)$$

which are the poles of `CWPQ`. If the argument is a lattice point, then b = `AMACH`(2), the largest floating-point number, is returned. If the argument has modulus greater than $10\varepsilon^{-1}$, then NaN (not a number) is returned. Here, ε = `AMACH`(4) is the machine precision.

Function `CWPQ` is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

CWPQD

This function evaluates the first derivative of the Weierstrass' $\wp$ function in the equianharmonic case for complex argument with unit period parallelogram.

Function Return Value

CWPQD — Complex function value. (Output)

Required Arguments

Z — Complex argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `CWPQD (Z)`

Specific: The specific interface names are `C_CWPQD` and `Z_CWPQD`.

FORTRAN 77 Interface

Complex: `CWPQD (Z)`

Double complex: The double complex name is `ZWPQD`.

Example

In this example, $\wp(0.25 + 0.14437567i)$ is computed and printed.

```
      USE CWPQD_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (0.25, 0.14437567)
      VALUE = CWPQD(Z)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, VALUE
99999 FORMAT (' CWPQD(', F6.3, ',', F6.3, ') = (', &
          F6.3, ',', F6.3, ')')
      END
```

Output

```
CWPQD( 0.250, 0.144) = ( 0.028,85.934)
```

Description

The Weierstrass' $\wp$ function, $\wp(z) = \wp(z \mid \omega, \omega')$, is an elliptic function of order two with periods 2ω and $2\omega'$ and a double pole at $z = 0$. `CWPQD(Z)` computes the derivative of $\wp(z \mid \omega, \omega')$ with

$$4\omega = 1 - i\sqrt{3} \text{ and } 4\omega' = 1 + i\sqrt{3}$$

`CWPQ`, page 176, computes $\wp(z \mid \omega, \omega')$.

The input argument is first reduced to the fundamental parallelogram of all z satisfying

$$-1/2 \le \Re z \le 1/2 \text{ and } -\sqrt{3}/4 \le \Im z \le \sqrt{3}/4$$

Then, a rational approximation is used.

All arguments are valid with the exception of the lattice points

$$z = m\left(1 - i\sqrt{3}\right) + n\left(1 + i\sqrt{3}\right)$$

which are the poles of `CWPQ`. If the argument is a lattice point, then b = `AMACH`(2), the largest floating-point number, is returned.

Function `CWPQD` is based on code by Eckhardt (1980). Also, see Eckhardt (1977).

EJSN

This function evaluates the Jacobi elliptic function sn(x, m).

Function Return Value

EJSN — Real or complex function value. (Output)

Required Arguments

X — Real or complex argument for which the function value is desired. (Input)

AM — Parameter of the elliptic function ($m = k^2$). (Input)

FORTRAN 90 Interface

Generic: `EJSN(X, AM)`

Specific: The specific interface names are `S_EJSN`, `D_EJSN`, `C_EJSN`, and `Z_EJSN`

FORTRAN 77 Interface

Single: `EJSN(X, AM)`

Double: The double precision name is `DEJSN`.

Complex: The complex name is `CEJSN`.

Double Complex: The double complex name is `ZEJSN`.

Example

In this example, sn(1.5, 0.5) is computed and printed.

```
      USE EJSN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       AM, VALUE, X
!                                 Compute
      AM    = 0.5
      X     = 1.5
      VALUE = EJSN(X, AM)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, AM, VALUE
99999 FORMAT (' EJSN(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
EJSN( 1.500, 0.500) = 0.968
```

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |X| is too large.

3	2	The result is accurate to less than one half precision because |`REAL (Z)`| is too large.
3	3	The result is accurate to less than one half precision because |`AIMAG (Z)`| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Description

The Jacobi elliptic function sn(x, m) = sin ϕ, where the amplitude ϕ is defined by the following:

$$x = \int_0^{\phi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{1/2}}$$

The function sn(x, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Additional Example

In this example, sn(1.5 + 0.3i, 0.5) is computed and printed.

```
      USE EJSN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       AM
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.5, 0.3)
      AM    = 0.5
      VALUE = EJSN(Z, AM)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, AM, VALUE
99999 FORMAT (' EJSN((', F6.3, ',', F6.3, '), ', F6.3, ') = (', &
           F6.3, ',', F6.3, ')')
      END
```

Output

```
EJSN(( 1.500, 0.300), 0.500) = ( 0.993, 0.054)
```

EJCN

This function evaluates the Jacobi elliptic function cn(x, m).

Function Return Value

EJCN — Real or complex function value. (Output)

Required Arguments

X — Real or complex argument for which the function value is desired. (Input)

AM — Parameter of the elliptic function ($m = k^2$). (Input)

FORTRAN 90 Interface

Generic: `EJCN(X, AM)`

Specific: The specific interface names are `S_EJCN`, `D_EJCN`, `C_EJCN`, and `Z_EJCN`.

FORTRAN 77 Interface

Single: `EJCN(X, AM)`

Double: The double precision name is `DEJCN`.

Complex: The complex name is `CEJCN`.

Double Complex: The double complex name is `ZEJCN`.

Example

In this example, cn(1.5, 0.5) is computed and printed.

```
      USE EJCN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       AM, VALUE, X
!                                 Compute
      AM    = 0.5
      X     = 1.5
      VALUE = EJCN(X, AM)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, AM, VALUE
99999 FORMAT (' EJCN(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
EJCN( 1.500, 0.500) = 0.250
```

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |`X`| is too large.
3	2	The result is accurate to less than one half precision because |`REAL (Z)`| is too large.
3	3	The result is accurate to less than one half precision because |`AIMAG (Z)`| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Description

The Jacobi elliptic function cn(x, m) = cos ϕ, where the amplitude ϕ is defined by the following:

$$x = \int_0^{\phi} \frac{d\theta}{\left(1 - m\sin^2\theta\right)^{1/2}}$$

The function cn(x, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Additional Example

In this example, cn(1.5 + 0.3i, 0.5) is computed and printed.

```
      USE EJCN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       AM
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.5, 0.3)
      AM    = 0.5
      VALUE = EJCN(Z, AM)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, AM, VALUE
99999 FORMAT (' EJCN((', F6.3, ',', F6.3, '), ', F6.3, ') = (', &
           F6.3, ',', F6.3, ')')
      END
```

Output

```
EJCN(( 1.500, 0.300), 0.500) = ( 0.251,-0.212)
```

EJDN

This function evaluates the Jacobi elliptic function dn(x, m).

Function Return Value

EJDN — Real or complex function value. (Output)

Required Arguments

X — Real or complex argument for which the function value is desired. (Input)

AM — Parameter of the elliptic function ($m = k^2$). (Input)

FORTRAN 90 Interface

Generic: `EJDN(X, AM)`

Specific: The specific interface names are `S_EJDN`, `D_EJDN`, `C_EJDN`, and `Z_EJDN`.

FORTRAN 77 Interface

Single: `EJDN(X, AM)`

Double: The double precision name is `DEJDN`.

Complex: The complex precision name is `CEJDN`.

Double Complex: The double complex precision name is `ZEJDN`.

Example

In this example, dn(1.5, 0.5) is computed and printed.

```
      USE EJDN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       AM, VALUE, X
!                                 Compute
      AM    = 0.5
      X     = 1.5
      VALUE = EJDN(X, AM)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, AM, VALUE
99999 FORMAT (' EJDN(', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
EJDN( 1.500, 0.500) = 0.729
```

Comments

Informational errors

Type	Code	
3	2	The result is accurate to less than one half precision because |`X`| is too large.
3	2	The result is accurate to less than one half precision because |`REAL (Z)`| is too large.
3	3	The result is accurate to less than one half precision because |`AIMAG (Z)`| is too large.
3	5	Landen transform did not converge. Result may not be accurate. This should never occur.

Description

The Jacobi elliptic function $\mathrm{dn}(x, m) = (1 - m \sin^2 \phi)^{1/2}$, where the amplitude ϕ is defined by the following:

$$x = \int_0^{\phi} \frac{d\theta}{\left(1 - m \sin^2 \theta\right)^{1/2}}$$

The function dn(x, m) is computed by first applying, if necessary, a Jacobi transformation so that the parameter, m, is between zero and one. Then, a descending Landen (Gauss) transform is applied until the parameter is small. The small parameter approximation is then applied.

Additional Example

In this example, dn(1.5 + 0.3i, 0.5) is computed and printed.

```
      USE EJDN_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       AM
      COMPLEX    VALUE, Z
!                                 Compute
      Z     = (1.5, 0.3)
      AM    = 0.5
      VALUE = EJDN(Z, AM)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) Z, AM, VALUE
99999 FORMAT (' EJDN((', F6.3, ',', F6.3, '), ', F6.3, ') = (', &
              F6.3, ',', F6.3, ')')
      END
```

Output

```
EJDN(( 1.500, 0.300), 0.500) = ( 0.714,-0.037)
```

Chapter 11: Probability Distribution Functions and Inverses

Routines

Usage Notes

Definitions and discussions of the terms basic to this chapter can be found in Johnson and Kotz (1969, 1970a, 1970b). These are also good references for the specific distributions.

In order to keep the calling sequences simple, whenever possible, the routines in this chapter are written for standard forms of statistical distributions. Hence, the number of parameters for any given distribution may be fewer than the number often associated with the distribution. For example, while a gamma distribution is often characterized by two parameters (or even a third, "location"), there is only one parameter that is necessary, the "shape." The "scale" parameter can be used to scale the variable to the *standard* gamma distribution. For another example, the functions relating to the normal distribution, `ANORDF` (page 206) and `ANORIN` (page 208), are for a normal distribution with mean equal to zero and variance equal to one. For other means and variances, it is very easy for the user to standardize the variables by subtracting the mean and dividing by the square root of the variance.

The *distribution function* for the (real, single-valued) random variable X is the function F defined for all real x by

$$F(x) = \text{Prob}(X \le x)$$

where Prob(·) denotes the probability of an event. The distribution function is often called the *cumulative distribution function* (CDF).

For distributions with finite ranges, such as the beta distribution, the CDF is 0 for values less than the left endpoint and 1 for values greater than the right endpoint. The routines in this chapter return the correct values for the distribution functions when values outside of the range of the random variable are input, but warning error conditions are set in these cases.

Discrete Random Variables

For discrete distributions, the function giving the probability that the random variable takes on specific values is called the *probability function*, defined by

$$p(x) = \text{Prob}(X = x)$$

The "`PR`" routines in this chapter evaluate probability functions.

The CDF for a discrete random variable is

$$F(x) = \sum_{A} p(k)$$

where A is the set such that $k \le x$. The "`DF`" routines in this chapter evaluate cumulative distributions functions. Since the distribution function is a step function, its inverse does not exist uniquely.

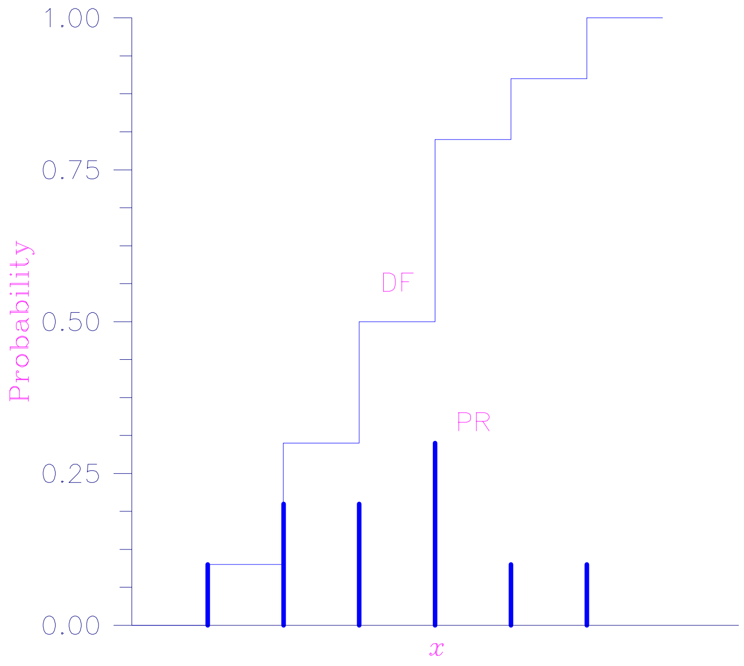


Figure 11-1 Discrete Random Variable

In the plot above, a routine like `BINPR` (page 191) in this chapter evaluates the individual probability, given *X*. A routine like `BINDF` (page 190) would evaluate the sum of the probabilities up to and including the probability at *X*.

Continuous Distributions

For continuous distributions, a probability function, as defined above, would not be useful because the probability of any given point is 0. For such distributions, the useful analog is the *probability density function* (PDF). The integral of the PDF is the probability over the interval; if the continuous random variable *X* has PDF *f*, then

$$\text{Prob}\left(a < X \le b\right) = \int_a^b f\left(x\right) dx$$

The relationship between the CDF and the PDF is

$$F\left(x\right) = \int_{-\infty}^{x} f\left(t\right) dt$$

as shown below.

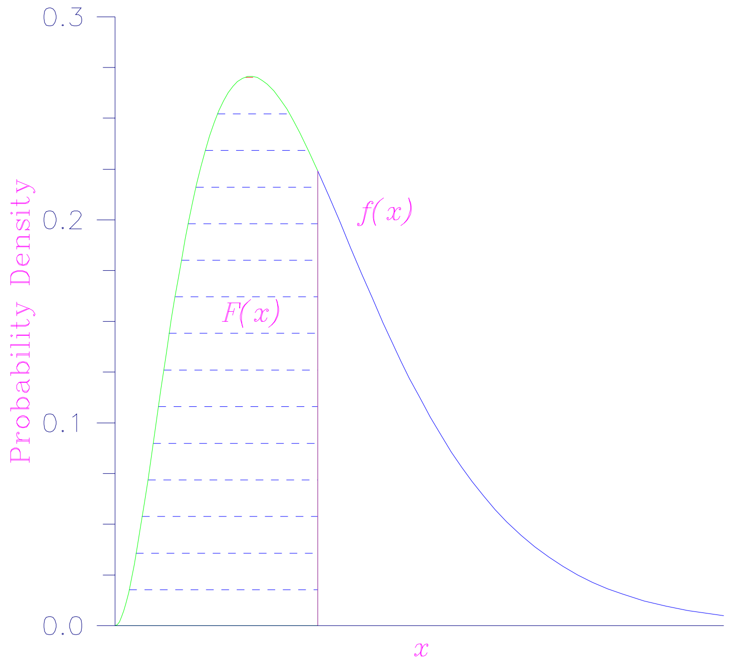


Figure 11-2   Probability Density Function

The “`DF`” routines for continuous distributions in this chapter evaluate cumulative distribution functions, just as the ones for discrete distributions.

For (absolutely) continuous distributions, the value of $F(x)$ uniquely determines x within the support of the distribution. The “`IN`” routines in this chapter compute the inverses of the distribution functions; that is, given $F(x)$ (called “`P`” for “probability”), a routine like `BETIN` (page 212) computes x. The inverses are defined only over the open interval (0, 1).

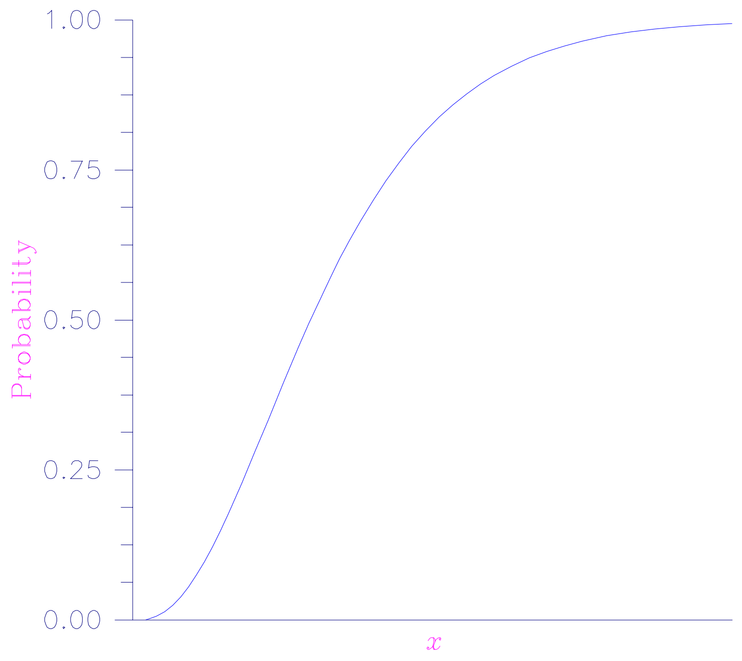


Figure 11-3 Cumulative Probability Distribution Function

There are two routines in this chapter that deal with general continuous distribution functions. The routine `GCDF` (page 233) computes a distribution function using values of the density function, and the routine `GCIN` (page 236) computes the inverse. These two routines may be useful when the user has an estimate of a probability density.

Additional Comments

Whenever a probability close to 1.0 results from a call to a distribution function or is to be input to an inverse function, it is often impossible to achieve good accuracy because of the nature of the representation of numeric values. In this case, it may be better to work with the complementary distribution function (one minus the distribution function). If the distribution is symmetric about some point (as the normal distribution, for example) or is reflective about some point (as the beta distribution, for example), the complementary distribution function has a simple relationship with the distribution function. For example, to evaluate the standard normal distribution at 4.0, using `ANORIN` (page 208) directly, the result to six places is 0.999968. Only two of those digits are really useful, however. A more useful result may be 1.000000 minus this value, which can be obtained to six significant figures as 3.16713E−05 by evaluating `ANORIN` at −4.0. For the normal distribution, the two values are related by $\Phi(x) = 1 - \Phi(-x)$, where $\Phi(\cdot)$ is the normal distribution function. Another example is the beta distribution with parameters 2 and 10. This distribution is skewed to the right; so evaluating `BETDF` at 0.7, we obtain 0.999953. A more precise result is obtained by evaluating `BETDF` with parameters 10 and 2 at 0.3. This yields 4.72392E−5. (In both of these examples, it is wise not to trust the last digit.)

Many of the algorithms used by routines in this chapter are discussed by Abramowitz and Stegun (1964). The algorithms make use of various expansions and recursive relationships, and often use different methods in different regions.

Cumulative distribution functions are defined for all real arguments; however, if the input to one of the distribution functions in this chapter is outside the range of the random variable, an error of Type 1 is issued, and the output is set to zero or one, as appropriate. A Type 1 error is of lowest severity, a “note;” and, by default, no printing or stopping of the program occurs. The other common errors that occur in the routines of this chapter are Type 2, “alert,” for a function value being set to zero due to underflow; Type 3, “warning,” for considerable loss of accuracy in the result returned; and Type 5, “terminal,” for incorrect and/ or inconsistent input, complete loss of accuracy in the result returned, or inability to represent the result (because of overflow). When a Type 5 error occurs, the result is set to NaN (not a number, also used as a missing value code, obtained by IMSL routine `AMACH`(6). (See the section “User Errors” in the Reference Material.)

BINDF

This function evaluates the binomial distribution function.

Function Return Value

BINDF — Function value, the probability that a binomial random variable takes a value less than or equal to `K`. (Output)
`BINDF` is the probability that `K` or fewer successes occur in `N` independent Bernoulli trials, each of which has a `P` probability of success.

Required Arguments

K — Argument for which the binomial distribution function is to be evaluated. (Input)

N — Number of Bernoulli trials. (Input)

P — Probability of success on each trial. (Input)

FORTRAN 90 Interface

Generic: `BINDF (K, N, P)`

Specific: The specific interface names are `S_BINDF` and `D_BINDF`.

FORTRAN 77 Interface

Single: `BINDF (K, N, P)`

Double: The double precision function name is `DBINDF`.

Example

Suppose X is a binomial random variable with $n = 5$ and $p = 0.95$. In this example, we find the probability that X is less than or equal to 3.

```
      USE IMSL_LIBRARIES
      INTEGER    K, N, NOUT
      REAL       P, PR
!
      CALL UMACH (2, NOUT)
      K  = 3
      N  = 5
      P  = 0.95
      PR = BINDF(K,N,P)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is less than or equal to 3 is ' &
            , F6.4)
      END
```

Output

```
The probability that X is less than or equal to 3 is 0.0226
```

Comments

Informational errors

Type	Code	
1	3	The input argument, `K`, is less than zero.
1	4	The input argument, `K`, is greater than the number of Bernoulli trials, `N`.

Description

Function `BINDF` evaluates the distribution function of a binomial random variable with parameters n and p. It does this by summing probabilities of the random variable taking on the specific values in its range. These probabilities are computed by the recursive relationship

$$\Pr\left(X = j\right) = \frac{\left(n+1-j\right)p}{j\left(1-p\right)}\Pr\left(X = j-1\right)$$

To avoid the possibility of underflow, the probabilities are computed forward from 0, if k is not greater than n times p, and are computed backward from n, otherwise. The smallest positive machine number, ε, is used as the starting value for summing the probabilities, which are rescaled by $(1 - p)^n\varepsilon$ if forward computation is performed and by $p^n\varepsilon$ if backward computation is done.

For the special case of $p = 0$, `BINDF` is set to 1; and for the case $p = 1$, `BINDF` is set to 1 if $k = n$ and to 0 otherwise.

BINPR

This function evaluates the binomial probability function.

Function Return Value

BINPR — Function value, the probability that a binomial random variable takes a value equal to `K`. (Output)

Required Arguments

K — Argument for which the binomial probability function is to be evaluated. (Input)

N — Number of Bernoulli trials. (Input)

P — Probability of success on each trial. (Input)

FORTRAN 90 Interface

Generic: `BINPR (K, N, P)`

Specific: The specific interface names are `S_BINPR` and `D_BINPR`.

FORTRAN 77 Interface

Single: `BINPR (K, N, P)`

Double: The double precision function name is `DBINPR`.

Example

Suppose X is a binomial random variable with $n = 5$ and $p = 0.95$. In this example, we find the probability that X is equal to 3.

```
      USE IMSL_LIBRARIES
      INTEGER    K, N, NOUT
      REAL       P, PR
!
      CALL UMACH (2, NOUT)
      K  = 3
      N  = 5
      P  = 0.95
      PR = BINPR(K,N,P)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is equal to 3 is ', F6.4)
      END
```

Output

```
The probability that X is equal to 3 is 0.0214
```

Comments

Informational errors
Type Code

1	3	The input argument, `K`, is less than zero.
1	4	The input argument, `K`, is greater than the number of Bernoulli trials, `N`.

Description

The function `BINPR` evaluates the probability that a binomial random variable with parameters *n* and *p* takes on the value *k*. It does this by computing probabilities of the random variable taking on the values in its range less than (or the values greater than) *k*. These probabilities are computed by the recursive relationship

$$\Pr(X = j) = \frac{(n+1-j)p}{j(1-p)} \Pr(X = j-1)$$

To avoid the possibility of underflow, the probabilities are computed forward from 0, if *k* is not greater than *n* times *p*, and are computed backward from *n*, otherwise. The smallest positive machine number, ε, is used as the starting value for computing the probabilities, which are rescaled by $(1-p)^n\varepsilon$ if forward computation is performed and by $p^n\varepsilon$ if backward computation is done.

For the special case of $p = 0$, `BINPR` is set to 0 if *k* is greater than 0 and to 1 otherwise; and for the case $p = 1$, `BINPR` is set to 0 if *k* is less than *n* and to 1 otherwise.

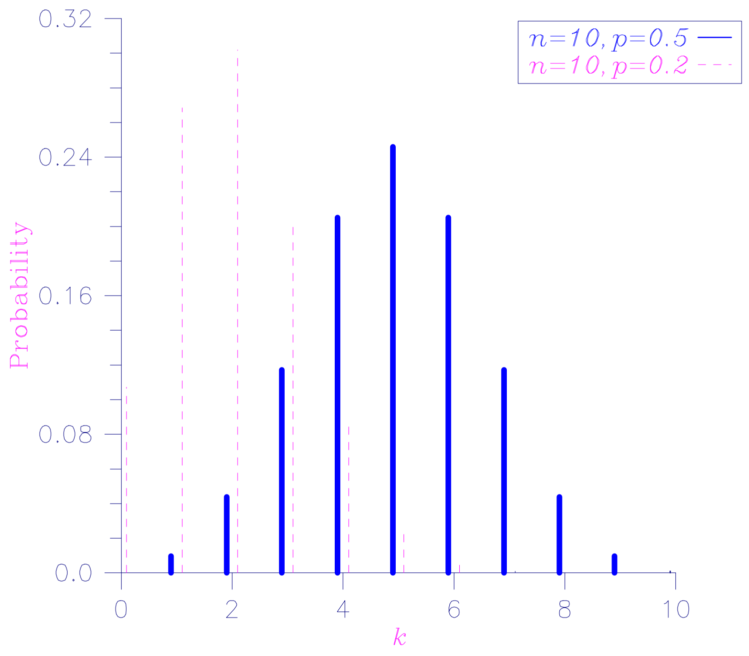


Figure 11-4 Binomial Probability Function

HYPDF

This function evaluates the hypergeometric distribution function.

Function Return Value

HYPDF — Function value, the probability that a hypergeometric random variable takes a value less than or equal to `K`. (Output)
`HYPDF` is the probability that `K` or fewer defectives occur in a sample of size `N` drawn from a lot of size `L` that contains `M` defectives.
See Comment 1.

Required Arguments

K — Argument for which the hypergeometric distribution function is to be evaluated. (Input)

N — Sample size. (Input)
`N` must be greater than zero and greater than or equal to `K`.

M — Number of defectives in the lot. (Input)

L — Lot size. (Input)
`L` must be greater than or equal to `N` and `M`.

FORTRAN 90 Interface

Generic: `HYPDF (K, N, M, L )`

Specific: The specific interface names are `S_HYPDF` and `D_HYPDF`.

FORTRAN 77 Interface

Single: `HYPDF (K, N, M, L)`

Double: The double precision function name is `DHYPDF`.

Example

Suppose X is a hypergeometric random variable with $n = 100$, $l = 1000$, and $m = 70$. In this example, we evaluate the distribution function at 7.

```
      USE IMSL_LIBRARIES
      INTEGER    K, L, M, N, NOUT
      REAL       DF
!
      CALL UMACH (2, NOUT)
      K  = 7
      N  = 100
      L  = 1000
      M  = 70
      DF = HYPDF(K,N,M,L)
      WRITE (NOUT,99999) DF
99999 FORMAT (' The probability that X is less than or equal to 7 is ' &
            , F6.4)
      END
```

Output

```
The probability that X is less than or equal to 7 is 0.5995
```

Comments

1. If the generic version of this function is used, the immediate result must be stored in a variable before use in an expression. For example:

   ```
   X = HYPDF(K, N, M, L)
   Y = SQRT(X)
   ```

   must be used rather than

   ```
   Y = SQRT(HYPDF(K, N, M, L))
   ```

   If this is too much of a restriction on the programmer, then the specific name can be used without this restriction.

2. Informational errors

Type	Code	
1	5	The input argument, `K`, is less than zero.
1	6	The input argument, `K`, is greater than the sample size.

Description

The function `HYPDF` evaluates the distribution function of a hypergeometric random variable with parameters *n*, *l*, and *m*. The hypergeometric random variable *X* can be thought of as the number of items of a given type in a random sample of size *n* that is drawn without replacement from a population of size *l* containing *m* items of this type. The probability function is

$$\Pr\left(X = j\right) = \frac{\binom{m}{j}\binom{l-m}{n-j}}{\binom{l}{n}} \quad \text{for } j = i,\ i+1,\ i+2,\ \ldots,\ \min\left(n,m\right)$$

where $i = \max(0, n - 1 + m)$.

If *k* is greater than or equal to *i* and less than or equal to min(*n*, *m*), `HYPDF` sums the terms in this expression for *j* going from *i* up to *k*. Otherwise, `HYPDF` returns 0 or 1, as appropriate. So, as to avoid rounding in the accumulation, `HYPDF` performs the summation differently depending on whether or not *k* is greater than the mode of the distribution, which is the greatest integer in $(m + 1)(n + 1)/(l + 2)$.

HYPPR

This function evaluates the hypergeometric probability function.

Function Return Value

HYPPR — Function value, the probability that a hypergeometric random variable takes a value equal to `K`. (Output)
`HYPPR` is the probability that exactly `K` defectives occur in a sample of size `N` drawn from a lot of size `L` that contains `M` defectives.
See Comment 1.

Required Arguments

K — Argument for which the hypergeometric probability function is to be evaluated. (Input)

N — Sample size. (Input)
`N` must be greater than zero and greater than or equal to `K`.

M — Number of defectives in the lot. (Input)

L — Lot size. (Input)
`L` must be greater than or equal to `N` and `M`.

FORTRAN 90 Interface

Generic: `HYPPR` (`K`, `N`, `M`, `L`)

Specific: The specific interface names are `S_HYPPR` and `D_HYPPR`.

FORTRAN 77 Interface

Single: `HYPPR (K, N, M, L)`

Double: The double precision function name is `DHYPPR`.

Example

Suppose X is a hypergeometric random variable with $n = 100$, $l = 1000$, and $m = 70$. In this example, we evaluate the probability function at 7.

```
      USE IMSL_LIBRARIES
      INTEGER    K, L, M, N, NOUT
      REAL       PR
!
      CALL UMACH (2, NOUT)
      K  = 7
      N  = 100
      L  = 1000
      M  = 70
      PR = HYPPR(K,N,M,L)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is equal to 7 is ', F6.4)
      END
```

Output

```
The probability that X is equal to 7 is 0.1628
```

Comments

1. If the generic version of this function is used, the immediate result must be stored in a variable before use in an expression. For example:

   ```
   X = HYPPR(K, N, M, L)
   Y = SQRT(X)
   ```

   must be used rather than

   ```
   Y = SQRT(HYPPR(K, N, M, L))
   ```

   If this is too much of a restriction on the programmer, then the specific name can be used without this restriction.

2. Informational errors

Type	Code	
1	5	The input argument, `K`, is less than zero.
1	6	The input argument, `K`, is greater than the sample size.

Description

The function `HYPPR` evaluates the probability function of a hypergeometric random variable with parameters *n*, *l*, and *m*. The hypergeometric random variable *X* can be thought of as the number of items of a given type in a random sample of size *n* that is drawn without replacement from a population of size *l* containing *m* items of this type. The probability function is

$$\Pr(X = k) = \frac{\binom{m}{k}\binom{l-m}{n-k}}{\binom{l}{n}} \quad \text{for } k = i,\ i+1,\ i+2,\ \ldots\ \min(n,m)$$

where $i = \max(0, n - l + m)$.

`HYPPR` evaluates the expression using log gamma functions.

POIDF

This function evaluates the Poisson distribution function.

Function Return Value

POIDF — Function value, the probability that a Poisson random variable takes a value less than or equal to `K`. (Output)

Required Arguments

K — Argument for which the Poisson distribution function is to be evaluated. (Input)

THETA — Mean of the Poisson distribution. (Input)
`THETA` must be positive.

FORTRAN 90 Interface

Generic: `POIDF (K, THETA)`

Specific: The specific interface names are `S_POIDF` and `D_POIDF`.

FORTRAN 77 Interface

Single: `POIDF (K, THETA)`

Double: The double precision function name is `DPOIDF`.

Example

Suppose *X* is a Poisson random variable with θ = 10. In this example, we evaluate the distribution function at 7.

```
      USE IMSL_LIBRARIES
      INTEGER    K, NOUT
      REAL       DF, THETA
!
      CALL UMACH (2, NOUT)
      K     = 7
      THETA = 10.0
      DF    = POIDF(K,THETA)
      WRITE (NOUT,99999) DF
99999 FORMAT (’ The probability that X is less than or equal to ’, &
            ’7 is ’, F6.4)
      END
```

Output

```
The probability that X is less than or equal to 7 is 0.2202
```

Comments

Informational error

Type	Code	
1	1	The input argument, `K`, is less than zero.

Description

The function `POIDF` evaluates the distribution function of a Poisson random variable with parameter `THETA`. `THETA`, which is the mean of the Poisson random variable, must be positive. The probability function (with θ = `THETA`) is

$$f(x) = e^{-\theta}\,\theta^{x}/x!, \text{ for } x = 0, 1, 2, \ldots$$

The individual terms are calculated from the tails of the distribution to the mode of the distribution and summed. `POIDF` uses the recursive relationship

$$f(x + 1) = f(x)\theta/(x + 1), \text{ for } x = 0, 1, 2, \ldots, k - 1$$

with $f(0) = e^{-\theta}$.

POIPR

This function evaluates the Poisson probability function.

Function Return Value

POIPR — Function value, the probability that a Poisson random variable takes a value equal to `K`. (Output)

Required Arguments

K — Argument for which the Poisson distribution function is to be evaluated. (Input)

THETA — Mean of the Poisson distribution. (Input)
`THETA` must be positive.

FORTRAN 90 Interface

Generic: `POIPR(K, THETA)`

Specific: The specific interface names are `S_POIPR` and `D_POIPR`.

FORTRAN 77 Interface

Single: `POIPR(K, THETA)`

Double: The double precision function name is `DPOIPR`.

Example

Suppose X is a Poisson random variable with θ = 10. In this example, we evaluate the probability function at 7.

```
      USE POIPR_INT
      USE UMACH_INT
      INTEGER    K, NOUT
      REAL       PR, THETA
!
      CALL UMACH (2, NOUT)
      K     = 7
      THETA = 10.0
      PR    = POIPR(K,THETA)
      WRITE (NOUT,99999) PR
99999 FORMAT (' The probability that X is equal to 7 is ', F6.4)
      END
```

Output

```
The probability that X is equal to 7 is 0.0901
```

Comments

Informational error

Type	Code	
1	1	The input argument, `K`, is less than zero.

Description

The function `POIPR` evaluates the probability function of a Poisson random variable with parameter `THETA`. `THETA`, which is the mean of the Poisson random variable, must be positive. The probability function (with θ = `THETA`) is

$$f(k) = e^{-\theta}\theta^k/k!, \text{ for } k = 0, 1, 2, \ldots$$

`POIPR` evaluates this function directly, taking logarithms and using the log gamma function.

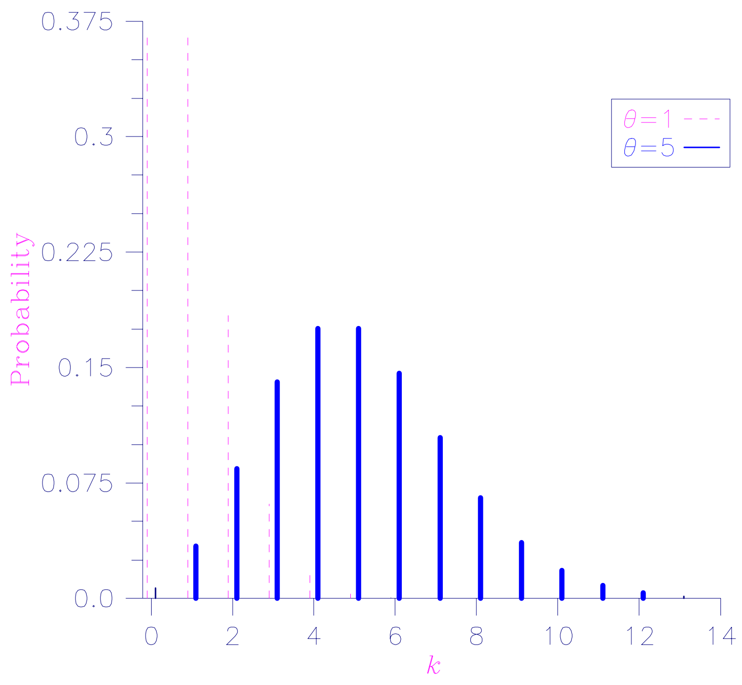


Figure 11-5 Poisson Probability Function

AKS1DF

This function evaluates the distribution function of the one-sided Kolmogorov-Smirnov goodness of fit D^+ or D^- test statistic based on continuous data for one sample.

Function Return Value

AKS1DF — The probability of a smaller `D`. (Output)

Required Arguments

NOBS — The total number of observations in the sample. (Input)

D — The D^+ or D^- test statistic. (Input)
`D` is the maximum positive difference of the empirical cumulative distribution function (CDF) minus the hypothetical CDF or the maximum positive difference of the hypothetical CDF minus the empirical CDF.

FORTRAN 90 Interface

Generic: `AKS1DF(NOBS, D)`

Specific: The specific interface names are `S_AKS1DF` and `D_AKS1DF`.

FORTRAN 77 Interface

Single: `AKS1DF(NOBS, D)`

Double: The double precision function name is `DKS1DF`.

Example

In this example, the exact one-sided probabilities for the tabled values of D^+ or D^-, given, for example, in Conover (1980, page 462), are computed. Tabled values at the 10% level of significance are used as input to `AKS1DF` for sample sizes of 5 to 50 in increments of 5. The last two tabled values are obtained using the asymptotic critical values of

$$1.07/\sqrt{\texttt{NOBS}}$$

The resulting probabilities should all be close to 0.90.

```
      USE AKS1DF_INT
      USE UMACH_INT
      INTEGER    I, NOBS, NOUT
      REAL       D(10)
!
      DATA D/0.447, 0.323, 0.266, 0.232, 0.208, 0.190, 0.177, 0.165, &
          0.160, 0.151/
!
      CALL UMACH (2, NOUT)
!
      DO 10  I=1, 10
         NOBS = 5*I
!
         WRITE (NOUT,99999) D(I), NOBS, AKS1DF(NOBS,D(I))
!
99999    FORMAT (' One-sided Probability for D = ', F8.3, ' with NOBS ' &
               , '= ', I2, ' is ', F8.4)
   10 CONTINUE
      END
```

Output

```
One-sided Probability for D =    0.447 with NOBS =  5 is   0.9000
One-sided Probability for D =    0.323 with NOBS = 10 is   0.9006
One-sided Probability for D =    0.266 with NOBS = 15 is   0.9002
One-sided Probability for D =    0.232 with NOBS = 20 is   0.9009
One-sided Probability for D =    0.208 with NOBS = 25 is   0.9002
One-sided Probability for D =    0.190 with NOBS = 30 is   0.8992
One-sided Probability for D =    0.177 with NOBS = 35 is   0.9011
One-sided Probability for D =    0.165 with NOBS = 40 is   0.8987
One-sided Probability for D =    0.160 with NOBS = 45 is   0.9105
One-sided Probability for D =    0.151 with NOBS = 50 is   0.9077
```

Comments

1. Workspace may be explicitly provided, if desired, by use of `AK21DF/DK21DF`. The reference is:

`AK2DF(NOBS, D, WK)`

The additional argument is:

WK — Work vector of length 3 * `NOBS` + 3 if `NOBS` ≤ 80. `WK` is not used if `NOBS` is greater than 80.

2. Informational errors

Type	Code	
1	2	Since the *D* test statistic is less than zero, the distribution function is zero at *D*.
1	3	Since the *D* test statistic is greater than one, the distribution function is one at *D*.

3. If `NOBS` ≤ 80, then exact one-sided probabilities are computed. In this case, on the order of `NOBS`^{2} operations are required. For `NOBS` > 80, approximate one-sided probabilities are computed. These approximate probabilities require very few computations.

4. An approximate two-sided probability for the `D` = max (D^+, D^-) statistic can be computed as twice the `AKS1DF` probability for `D` (minus one, if the probability from `AKS1DF` is greater than 0.5).

Description

Routine `AKS1DF` computes the cumulative distribution function (CDF) for the one-sided Kolmogorov-Smirnov one-sample D^+ or D^- statistic when the theoretical CDF is strictly continuous. Exact probabilities are computed according to a method given by Conover (1980, page 350) for sample sizes of 80 or less. For sample sizes greater than 80, the asympotic methods discussed by Conover are used.

Let $F(x)$ denote the theoretical distribution function, and let $S_n(x)$ denote the empirical distribution function obtained from a sample of size `NOBS`. Then, the D^+ statistic is computed as

$$D^+ = \sup_x \left[F(x) - S_n(x) \right]$$

while the one-sided D^- statistic is computed as

$$D^- = \sup_x \left[S_n(x) - F(x) \right]$$

Programming Notes

Routine `AKS1DF` requires on the order of `NOBS`^{2} operations to compute the exact probabilities, where an operation consists of taking ten or so logarithms. Because so much computation is occurring within each “operation,” `AKS1DF` is much slower than its two-sample counterpart, IMSL function `AKS2DF` (page 204).

AKS2DF

This function evaluates the distribution function of the Kolmogorov-Smirnov goodness of fit *D* test statistic based on continuous data for two samples.

Function Return Value

AKS2DF — The probability of a smaller `D`. (Output)

Required Arguments

NOBSX — The total number of observations in the first sample. (Input)

NOBSY — The total number of observations in the second sample. (Input)

D — The *D* test statistic. (Input)
`D` is the maximum absolute difference between empirical cumulative distribution functions (CDFs) of the two samples.

FORTRAN 90 Interface

Generic: `AKS2DF(NOBSX, NOBSY, D)`

Specific: The specific interface names are `S_AKS2DF` and `D_AKS2DF`.

FORTRAN 77 Interface

Single: `AKS2DF(NOBSX, NOBSY, D)`

Double: The double precision function name is `DKS2DF`.

Example

Function `AKS2DF` is used to compute the probability of a smaller *D* statistic for a variety of sample sizes using values close to the 0.95 probability value.

```
      USE AKS2DF_INT
      USE UMACH_INT
      INTEGER    I, NOBSX(10), NOBSY(10), NOUT
      REAL       D(10)
!
      DATA NOBSX/5, 20, 40, 70, 110, 200, 200, 200, 100, 100/
      DATA NOBSY/10, 10, 10, 10, 10, 20, 40, 60, 80, 100/
      DATA D/0.7, 0.55, 0.475, 0.4429, 0.4029, 0.2861, 0.2113, 0.1796, &
          0.18, 0.18/
!
      CALL UMACH (2, NOUT)
!
      DO 10  I=1, 10
!
```

```
         WRITE (NOUT,99999) D(I), NOBSX(I), NOBSY(I), &
                          AKS2DF(NOBSX(I),NOBSY(I),D(I))
!
99999    FORMAT (’ Probability for D = ’, F5.3, ’ with NOBSX = ’, I3, &
               ’ and NOBSY = ’, I3, ’ is ’, F9.6, ’.’)
   10 CONTINUE
      END
```

Output

```
Probability for D = 0.700 with NOBSX =   5 and NOBSY =  10 is  0.980686.
Probability for D = 0.550 with NOBSX =  20 and NOBSY =  10 is  0.987553.
Probability for D = 0.475 with NOBSX =  40 and NOBSY =  10 is  0.972423.
Probability for D = 0.443 with NOBSX =  70 and NOBSY =  10 is  0.961646.
Probability for D = 0.403 with NOBSX = 110 and NOBSY =  10 is  0.928667.
Probability for D = 0.286 with NOBSX = 200 and NOBSY =  20 is  0.921126.
Probability for D = 0.211 with NOBSX = 200 and NOBSY =  40 is  0.917110.
Probability for D = 0.180 with NOBSX = 200 and NOBSY =  60 is  0.914520.
Probability for D = 0.180 with NOBSX = 100 and NOBSY =  80 is  0.908185.
Probability for D = 0.180 with NOBSX = 100 and NOBSY = 100 is  0.946098.
```

Comments

1. Workspace may be explicitly provided, if desired, by use of `AK22DF/DK22DF`. The reference is:

   `AK22DF(NOBSX, NOBSY, D, WK)`

   The additional argument is

   WK — Work vector of length max(`NOBSX`, `NOBSY`) + 1.

2. Informational errors

Type	Code	
1	2	Since the *D* test statistic is less than zero, then the distribution function is zero at `D`.
1	3	Since the *D* test statistic is greater than one, then the distribution function is one at `D`.

Description

Function `AKS2DF` computes the cumulative distribution function (CDF) for the two-sided Kolmogorov-Smirnov two-sample *D* statistic when the theoretical CDF is strictly continuous. Exact probabilities are computed according to a method given by Kim and Jennrich (1973). Approximate asymptotic probabilities are computed according to methods also given in this reference.

Let $F_n(x)$ and $G_m(x)$ denote the empirical distribution functions for the two samples, based on n = `NOBSX` and m = `NOBSY` observations. Then, the *D* statistic is computed as

$$D = \sup_x \left| F_n(x) - G_m(x) \right|$$

Programming Notes

Function AKS2DF requires on the order of NOBSX * NOBSY operations to compute the exact probabilities, where an operation consists of an addition and a multiplication. For NOBSX * NOBSY less than 10000, the exact probability is computed. If this is not the case, then the Smirnov approximation discussed by Kim and Jennrich is used if the minimum of NOBSX and NOBSY is greater than ten percent of the maximum of NOBSX and NOBSY, or if the minimum is greater than 80. Otherwise, the Kolmogorov approximation discussed by Kim and Jennrich is used.

ANORDF

This function evaluates the standard normal (Gaussian) distribution function.

Function Return Value

ANORDF — Function value, the probability that a normal random variable takes a value less than or equal to X. (Output)

Required Arguments

X — Argument for which the normal distribution function is to be evaluated. (Input)

FORTRAN 90 Interface

Generic: ANORDF (X)

Specific: The specific interface names are S_ANORDF and D_ANORDF.

FORTRAN 77 Interface

Single: ANORDF (X)

Double: The double precision function name is DNORDF.

Example

Suppose *X* is a normal random variable with mean 100 and variance 225. In this example, we find the probability that *X* is less than 90, and the probability that *X* is between 105 and 110.

```
      USE ANORDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       P, X1, X2
!
      CALL UMACH (2, NOUT)
      X1 = (90.0-100.0)/15.0
      P  = ANORDF(X1)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that X is less than 90 is ', F6.4)
```

```
      X1 = (105.0-100.0)/15.0
      X2 = (110.0-100.0)/15.0
      P  = ANORDF(X2) - ANORDF(X1)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is between 105 and 110 is ', &
            F6.4)
      END
```

Output

```
The probability that X is less than 90 is 0.2525
The probability that X is between 105 and 110 is 0.1169
```

Description

Function `ANORDF` evaluates the distribution function, Φ, of a standard normal (Gaussian) random variable, that is,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

The value of the distribution function at the point *x* is the probability that the random variable takes a value less than or equal to *x*.

The standard normal distribution (for which `ANORDF` is the distribution function) has mean of 0 and variance of 1. The probability that a normal random variable with mean μ and variance σ^2 is less than *y* is given by `ANORDF` evaluated at $(y - \mu)/\sigma$.

Φ(*x*) is evaluated by use of the complementary error function, erfc. (See `ERFC` in Chapter 5, "Error Funtions and Related Functions" of this manual.)) The relationship is:

$$\Phi(x) = \operatorname{erfc}\left(-x/\sqrt{2.0}\right)/2$$

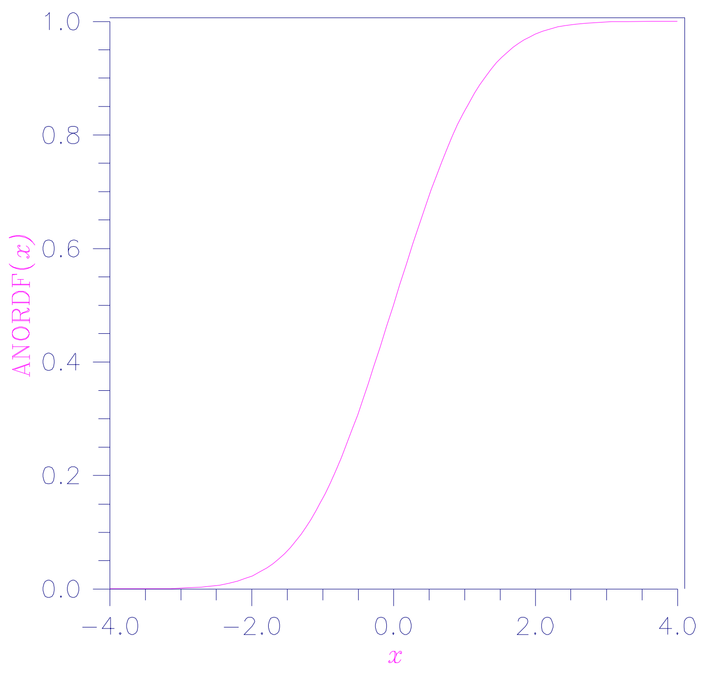


Figure 11-6   Standard Normal Distribution Function

ANORIN

This function evaluates the inverse of the standard normal (Gaussian) distribution function.

Function Return Value

ANORIN — Function value.   (Output)
The probability that a standard normal random variable takes a value less than or equal to `ANORIN` is `P`.

Required Arguments

P — Probability for which the inverse of the normal distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

FORTRAN 90 Interface

Generic: `ANORIN (P)`

Specific: The specific interface names are `S_ANORIN` and `D_ANORIN`.

FORTRAN 77 Interface

Single: ANORIN (P)

Double: The double precision function name is DNORIN.

Example

In this example, we compute the point such that the probability is 0.9 that a standard normal random variable is less than or equal to this point.

```
      USE ANORIN_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       P, X
!
      CALL UMACH (2, NOUT)
      P = 0.9
      X = ANORIN(P)
      WRITE (NOUT,99999) X
99999 FORMAT (' The 90th percentile of a standard normal is ', F6.4)
      END
```

Output

```
The 90th percentile of a standard normal is 1.2816
```

Description

Function ANORIN evaluates the inverse of the distribution function, Φ, of a standard normal (Gaussian) random variable, that is, ANORIN(P) = $\Phi^{-1}(p)$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

The value of the distribution function at the point *x* is the probability that the random variable takes a value less than or equal to *x*. The standard normal distribution has a mean of 0 and a variance of 1.

References used to design this routine include Hart et al. (1968), Kinnucan and Kuki (1968), and Strecok (1968).

BETDF

This function evaluates the beta probability distribution function.

Function Return Value

BETDF — Probability that a random variable from a beta distribution having parameters PIN and QIN will be less than or equal to X. (Output)

Required Arguments

X — Argument for which the beta distribution function is to be evaluated. (Input)

PIN — First beta distribution parameter. (Input)
`PIN` must be positive.

QIN — Second beta distribution parameter. (Input)
`QIN` must be positive.

FORTRAN 90 Interface

Generic: `BETDF(X, PIN, QIN)`

Specific: The specific interface names are `S_BETDF` and `D_BETDF`.

FORTRAN 77 Interface

Single: `BETDF(X, PIN, QIN)`

Double: The double precision function name is `DBETDF`.

Example

Suppose *X* is a beta random variable with parameters 12 and 12. (*X* has a symmetric distribution.) In this example, we find the probability that *X* is less than 0.6 and the probability that *X* is between 0.5 and 0.6. (Since *X* is a symmetric beta random variable, the probability that it is less than 0.5 is 0.5.)

```
      USE BETDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       P, PIN, QIN, X
!
      CALL UMACH (2, NOUT)
      PIN = 12.0
      QIN = 12.0
      X   = 0.6
      P   = BETDF(X,PIN,QIN)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that X is less than 0.6 is ', F6.4)
      X = 0.5
      P = P - BETDF(X,PIN,QIN)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is between 0.5 and 0.6 is ', &
            F6.4)
      END
```

Output

```
The probability that X is less than 0.6 is 0.8364
The probability that X is between 0.5 and 0.6 is 0.3364
```

Comments

Informational errors

Type	Code	
1	1	Since the input argument `X` is less than or equal to zero, the distribution function is equal to zero at `X`.
1	2	Since the input argument `X` is greater than or equal to one, the distribution function is equal to one at `X`.

Description

Function `BETDF` evaluates the distribution function of a beta random variable with parameters `PIN` and `QIN`. This function is sometimes called the *incomplete beta ratio* and, with p = `PIN` and q = `QIN`, is denoted by $I_x(p, q)$. It is given by

$$I_x(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_0^x t^{p-1}(1-t)^{q-1}\,dt$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function $I_x(p, q)$ is the probability that the random variable takes a value less than or equal to x.

The integral in the expression above is called the *incomplete beta function* and is denoted by $\beta_x(p, q)$. The constant in the expression is the reciprocal of the *beta function* (the incomplete function evaluated at one) and is denoted by $\beta(p, q)$.

Function `BETDF` uses the method of Bosten and Battiste (1974b).

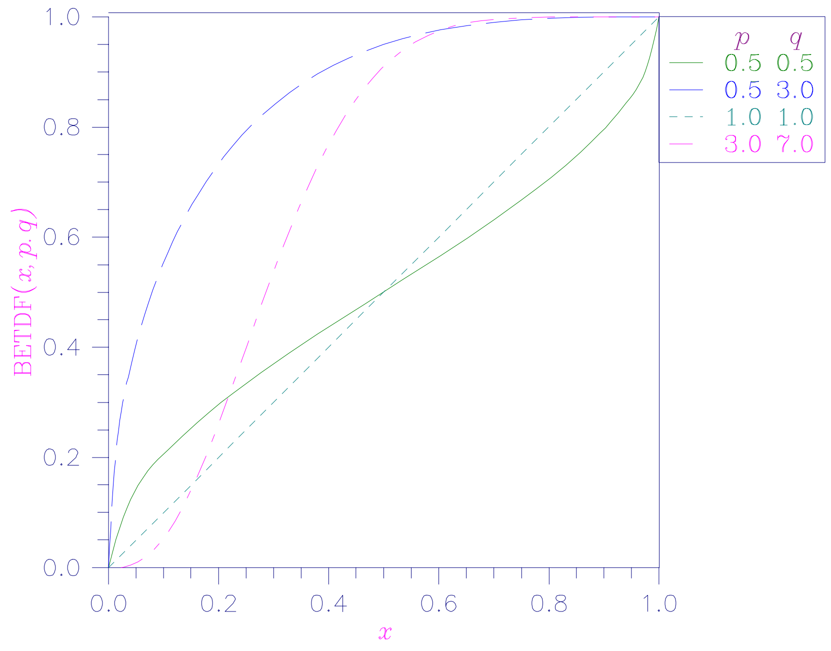


Figure 11-7 Beta Distribution Function

BETIN

This function evaluates the inverse of the beta distribution function.

Function Return Value

BETIN — Function value. (Output)
The probability that a beta random variable takes a value less than or equal to `BETIN` is `P`.

Required Arguments

P — Probability for which the inverse of the beta distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

PIN — First beta distribution parameter. (Input)
`PIN` must be positive.

QIN — Second beta distribution parameter. (Input)
`QIN` must be positive.

FORTRAN 90 Interface

Generic: `BETIN(P, PIN, QIN)`

Specific: The specific interface names are `S_BETIN` and `D_BETIN`.

FORTRAN 77 Interface

Single: `BETIN(P, PIN, QIN)`

Double: The double precision function name is `DBETIN`.

Example

Suppose X is a beta random variable with parameters 12 and 12. (X has a symmetric distribution.) In this example, we find the value x_0 such that the probability that $X \le x_0$ is 0.9.

```
      USE BETIN_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       P, PIN, QIN, X
!
      CALL UMACH (2, NOUT)
      PIN = 12.0
      QIN = 12.0
      P   = 0.9
      X   = BETIN(P,PIN,QIN)
      WRITE (NOUT,99999) X
99999 FORMAT (' X is less than ', F6.4, ' with probability 0.9.')
      END
```

Output

```
X is less than 0.6299 with probability 0.9.
```

Comments

Informational error

Type	Code	
3	1	The value for the inverse Beta distribution could not be found in 100 iterations. The best approximation is used.

Description

The function `BETIN` evaluates the inverse distribution function of a beta random variable with parameters `PIN` and `QIN`, that is, with P = `P`, p = `PIN`, and q = `QIN`; it determines x (= `BETIN`(`P`, `PIN`, `QIN`)), such that

$$P = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_0^x t^{p-1} (1-t)^{q-1} \, dt$$

where $\Gamma(\cdot)$ is the gamma function. The probability that the random variable takes a value less than or equal to x is P.

BNRDF

This function evaluates the bivariate normal distribution function.

Function Return Value

BNRDF — Function value, the probability that a bivariate normal random variable with correlation `RHO` takes a value less than or equal to `X` and less than or equal to `Y`. (Output)

Required Arguments

X — One argument for which the bivariate normal distribution function is to be evaluated. (Input)

Y — The other argument for which the bivariate normal distribution function is to be evaluated. (Input)

RHO — Correlation coefficient. (Input)

FORTRAN 90 Interface

Generic: `BNRDF(X, Y, RHO)`

Specific: The specific interface names are S_BNRDF and D_BNRDF.

FORTRAN 77 Interface

Single: `BNRDF(X, Y, RHO)`

Double: The double precision function name is DBNRDF.

Example

Suppose (X, Y) is a bivariate normal random variable with mean (0, 0) and variance-covariance matrix

$$\begin{pmatrix} 1.0 & 0.9 \\ 0.9 & 1.0 \end{pmatrix}$$

In this example, we find the probability that X is less than –2.0 and Y is less than 0.0.

```
      USE BNRDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       P, RHO, X, Y
!
      CALL UMACH (2, NOUT)
      X   = -2.0
      Y   = 0.0
      RHO = 0.9
      P   = BNRDF(X,Y,RHO)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is less than -2.0  and Y ', &
            'is less than 0.0 is ', F6.4)
      END
```

Output

```
The probability that X is less than -2.0 and Y is less than 0.0 is 0.0228
```

Description

Function `BNRDF` evaluates the distribution function F of a bivariate normal distribution with means of zero, variances of one, and correlation of `RHO`, that is, with ρ = `RHO`, and $|\rho| < 1$,

$$F(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left(-\frac{u^2 - 2\rho uv + v^2}{2\left(1-\rho^2\right)} \right) du\, dv$$

To determine the probability that $U \le u_0$ and $V \le v_0$, where $(U, V)^T$ is a bivariate normal random variable with mean $\mu = (\mu_U, \mu_V)^T$ and variance-covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_U^2 & \sigma_{UV} \\ \sigma_{UV} & \sigma_V^2 \end{pmatrix}$$

transform $(U, V)^T$ to a vector with zero means and unit variances. The input to BNRDF would be X = $(u_0 - \mu_U)/\sigma_U$, Y = $(v_0 - \mu_V) = \sigma_V$, and $\rho = \sigma_{UV}/(\sigma_U\sigma_V)$.

Function BNRDF uses the method of Owen (1962, 1965). For $|\rho| = 1$, the distribution function is computed based on the univariate statistic, $Z = \min(x, y)$, and on the normal distribution function ANORDF (page 206).

See Cooper (1968) for more information on the algorithm used.

CHIDF

This function evaluates the chi-squared distribution function.

Function Return Value

CHIDF — Function value, the probability that a chi-squared random variable takes a value less than or equal to CHSQ. (Output)

Required Arguments

CHSQ — Argument for which the chi-squared distribution function is to be evaluated. (Input)

DF — Number of degrees of freedom of the chi-squared distribution. (Input)
DF must be greater than or equal to 0.5.

FORTRAN 90 Interface

Generic: CHIDF(CHSQ, DF)

Specific: The specific interface names are S_CHIDF and D_CHIDF.

FORTRAN 77 Interface

Single: CHIDF(CHSQ, DF)

Double: The double precision function name is DCHIDF.

Example

Suppose X is a chi-squared random variable with 2 degrees of freedom. In this example, we find the probability that X is less than 0.15 and the probability that X is greater than 3.0.

```
      USE UMACH_INT
      USE CHIDF_INT
      INTEGER    NOUT
      REAL       CHSQ, DF, P
!
      CALL UMACH (2, NOUT)
      DF   = 2.0
      CHSQ = 0.15
      P    = CHIDF(CHSQ,DF)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that chi-squared with 2 df is less ', &
             'than 0.15 is ', F6.4)
      CHSQ = 3.0
      P    = 1.0 - CHIDF(CHSQ,DF)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that chi-squared with 2 df is greater ', &
             'than 3.0 is ', F6.4)
      END
```

Output

```
The probability that chi-squared with 2 df is less than 0.15 is 0.0723
The probability that chi-squared with 2 df is greater than 3.0 is 0.2231
```

Comments

Informational errors

Type	Code	
1	1	Since the input argument, `CHSQ`, is less than zero, the distribution function is zero at `CHSQ`.
2	3	The normal distribution is used for large degrees of freedom. However, it has produced underflow. Therefore, the probability, `CHIDF`, is set to zero.

Description

Function `CHIDF` evaluates the distribution function, *F*, of a chi-squared random variable with `DF` degrees of freedom, that is, with ν = `DF`, and *x* = `CHSQ`,

$$F(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} \int_0^x e^{-t/2} t^{\nu/2-1} dt$$

where Γ(·) is the gamma function. The value of the distribution function at the point *x* is the probability that the random variable takes a value less than or equal to *x*.

For ν > 65, `CHIDF` uses the Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) to the normal distribution, and routine `ANORDF` (page 206) is used to evaluate the normal distribution function.

For ν ≤ 65, `CHIDF` uses series expansions to evaluate the distribution function. If $x < \max(\nu/2, 26)$, `CHIDF` uses the series 6.5.29 in Abramowitz and Stegun (1964); otherwise, it uses the asymptotic expansion 6.5.32 in Abramowitz and Stegun.

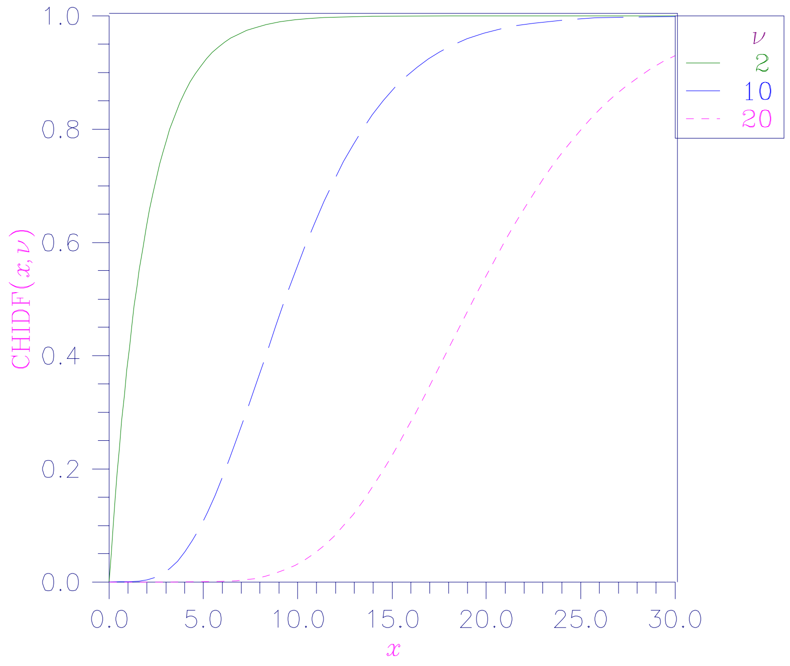


Figure 11-8 Chi-Squared Distribution Function

CHIIN

This function evaluates the inverse of the chi-squared distribution function.

Function Return Value

CHIIN — Function value. (Output)
The probability that a chi-squared random variable takes a value less than or equal to `CHIIN` is `P`.

Required Arguments

P — Probability for which the inverse of the chi-squared distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

DF — Number of degrees of freedom of the chi-squared distribution. (Input)
`DF` must be greater than or equal to 0.5.

FORTRAN 90 Interface

Generic: `CHIIN(P, DF)`

Specific: The specific interface names are `S_CHIIN` and `D_CHIIN`.

FORTRAN 77 Interface

Single: `CHIIN(P, DF)`

Double: The double precision function name is `DCHIIN`.

Example

In this example, we find the 99-th percentage points of a chi-squared random variable with 2 degrees of freedom and of one with 64 degrees of freedom.

```
      USE CHIIN_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       DF, P, X
!
      CALL UMACH (2, NOUT)
      P  = 0.99
      DF = 2.0
      X  = CHIIN(P,DF)
      WRITE (NOUT,99998) X
99998 FORMAT (' The 99-th percentage point of chi-squared with  2 df ' &
             , 'is ', F7.3)
      DF = 64.0
      X  = CHIIN(P,DF)
      WRITE (NOUT,99999) X
99999 FORMAT (' The 99-th percentage point of chi-squared with 64 df ' &
            , 'is ', F7.3)
      END
```

Output

```
The 99-th percentage point of chi-squared with 2 df is 9.210
The 99-th percentage point of chi-squared with 64 df is 93.217
```

Comments

Informational errors

Type	Code	
4	1	Over 100 iterations have occurred without convergence. Convergence is assumed.

Description

Function `CHIIN` evaluates the inverse distribution function of a chi-squared random variable with `DF` degrees of freedom; that is, with P = `P` and ν = `DF`, it determines x (= `CHIIN(P, DF)`), such that

$$P = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\int_0^x e^{-t/2}t^{\nu/2-1}dt$$

where Γ(·) is the gamma function. The probability that the random variable takes a value less than or equal to x is P.

For $\nu < 40$, `CHIIN` uses bisection (if $\nu \le 2$ or $P > 0.98$) or regula falsi to find the point at which the chi-squared distribution function is equal to P. The distribution function is evaluated using routine `CHIDF` (page 215).

For $40 \le \nu < 100$, a modified Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.18) to the normal distribution is used, and routine `ANORIN` (page 208) is used to evaluate the inverse of the normal distribution function. For $\nu \ge 100$, the ordinary Wilson-Hilferty approximation (Abramowitz and Stegun 1964, equation 26.4.17) is used.

CSNDF

This function evaluates the noncentral chi-squared distribution function.

Function Return Value

CSNDF — Function value, the probability that a noncentral chi-squared random variable takes a value less than or equal to `CHSQ`. (Output)

Required Arguments

CHSQ — Argument for which the noncentral chi-squared distribution function is to be evaluated. (Input)

DF — Number of degrees of freedom of the noncentral chi-squared distribution. (Input)
`DF` must be greater than or equal to 0.5 and less than or equal to 200,000.

ALAM — The noncentrality parameter. (Input)
`ALAM` must be nonnegative, and `ALAM` + `DF` must be less than or equal to 200,000.

FORTRAN 90 Interface

Generic: `CSNDF(CHSQ, DF, ALAM)`

Specific: The specific interface names are `S_CSNDF` and `D_CSNDF`.

FORTRAN 77 Interface

Single: `CSNDF(CHSQ, DF, ALAM)`

Double: The double precision function name is `DCSNDF`.

Example

In this example, CSNDF is used to compute the probability that a random variable that follows the noncentral chi-squared distribution with noncentrality parameter of 1 and with 2 degrees of freedom is less than or equal to 8.642.

```
      USE CSNDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       ALAM, CHSQ, DF, P
!
      CALL UMACH (2, NOUT)
      DF   = 2.0
      ALAM = 1.0
      CHSQ = 8.642
      P    = CSNDF(CHSQ,DF,ALAM)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that a noncentral chi-squared random', &
            /, ' variable with 2 df and noncentrality 1.0 is less', &
            /, ' than 8.642 is ', F5.3)
      END
```

Output

```
The probability that a noncentral chi-squared random
variable with 2 df and noncentrality 1.0 is less
than 8.642 is 0.950
```

Comments

1. Informational errors

Type	Code	
1	1	Since the input argument, CHSQ, is less than or equal to zero, the distribution function is zero at CHSQ.
3	2	Convergence was not obtained. The best approximation to the probability is returned.

2. This subroutine sums terms of an infinite series of central chi-squared distribution functions weighted by Poisson terms. Summing terminates when either the current term is less than 10 * AMACH(4) times the current sum or when 1000 terms have been accumulated. In the latter case, a warning error is issued.

Description

Function CSNDF evaluates the distribution function of a noncentral chi-squared random variable with DF degrees of freedom and noncentrality parameter ALAM; that is, with ν = DF, λ = ALAM, and x = CHSQ,

$$\mathrm{CSNDF}(x) = \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^i}{i!} \int_0^x \frac{t^{(\nu+2i)/2-1} e^{-t/2}}{2^{(\nu+2i)/2}\Gamma\left(\frac{\nu+2i}{2}\right)} dt$$

where $\Gamma(\cdot)$ is the gamma function. This is a series of central chi-squared distribution functions with Poisson weights. The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x.

The noncentral chi-squared random variable can be defined by the distribution function above, or alternatively and equivalently, as the sum of squares of independent normal random variables. If Y_i have independent normal distributions with means μ_i and variances equal to one and

$$X = \sum_{i=1}^{n} Y_i^2$$

then X has a noncentral chi-squared distribution with n degrees of freedom and noncentrality parameter equal to

$$\sum_{i=1}^{n} \mu_i^2$$

With a noncentrality parameter of zero, the noncentral chi-squared distribution is the same as the chi-squared distribution.

Function `CSNDF` determines the point at which the Poisson weight is greatest, and then sums forward and backward from that point, terminating when the additional terms are sufficiently small or when a maximum of 1000 terms have been accumulated. The recurrence relation 26.4.8 of Abramowitz and Stegun (1964) is used to speed the evaluation of the central chi-squared distribution functions.

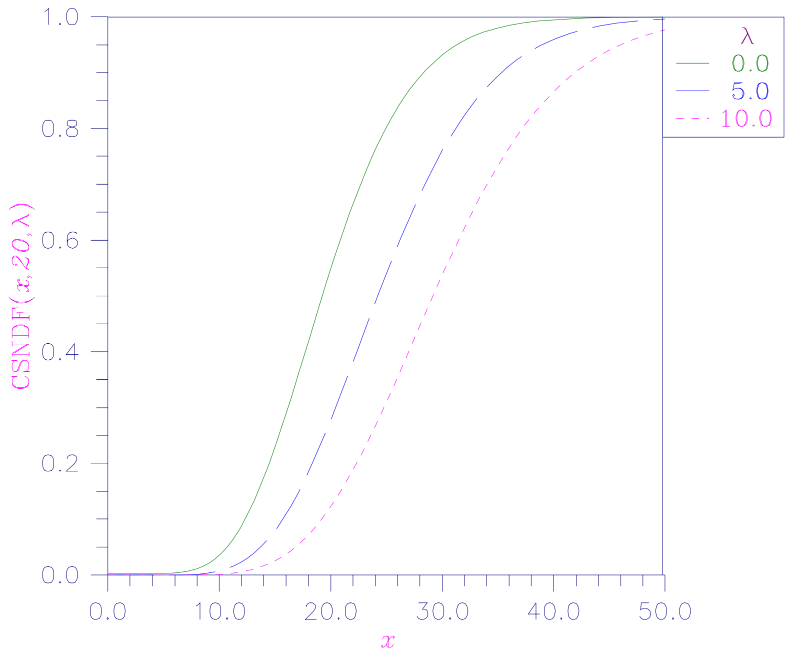


Figure 11-9 Noncentral Chi-squared Distribution Function

FDF

This function evaluates the *F* distribution function.

Function Return Value

FDF — Function value, the probability that an *F* random variable takes a value less than or equal to the input `F`. (Output)

Required Arguments

F — Argument for which the *F* distribution function is to be evaluated. (Input)

DFN — Numerator degrees of freedom. (Input)
`DFN` must be positive.

DFD — Denominator degrees of freedom. (Input)
`DFD` must be positive.

FORTRAN 90 Interface

Generic: `FDF(F, DFN, DFD)`

Specific: The specific interface names are `S_FDF` and `D_FDF`.

FORTRAN 77 Interface

Single: `FDF(F, DFN, DFD)`

Double: The double precision function name is `DFDF`.

Example

In this example, we find the probability that an *F* random variable with one numerator and one denominator degree of freedom is greater than 648.

```
      USE FDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       DFD, DFN, F, P
!
      CALL UMACH (2, NOUT)
      F   = 648.0
      DFN = 1.0
      DFD = 1.0
      P   = 1.0 - FDF(F,DFN,DFD)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that an F(1,1) variate is greater ', &
            'than 648 is ', F6.4)
      END
```

Output

```
The probability that an F(1,1) variate is greater than 648 is 0.0250
```

Comments

Informational error

Type	Code	
1	3	Since the input argument `F` is not positive, the distribution function is zero at `F`.

Description

Function `FDF` evaluates the distribution function of a Snedecor's *F* random variable with `DFN` numerator degrees of freedom and `DFD` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using the routine `BETDF` (page 209). If *X* is an *F* variate with ν_1 and ν_2 degrees of freedom and $Y = \nu_1 X/(\nu_2 + \nu_1 X)$, then *Y* is a beta variate with parameters $p = \nu_1/2$ and $q = \nu_2/2$. The function `FDF` also uses a relationship between *F* random variables that can be expressed as follows:

```
FDF(X, DFN, DFD) = 1.0 - FDF(1.0/X, DFD, DFN)
```

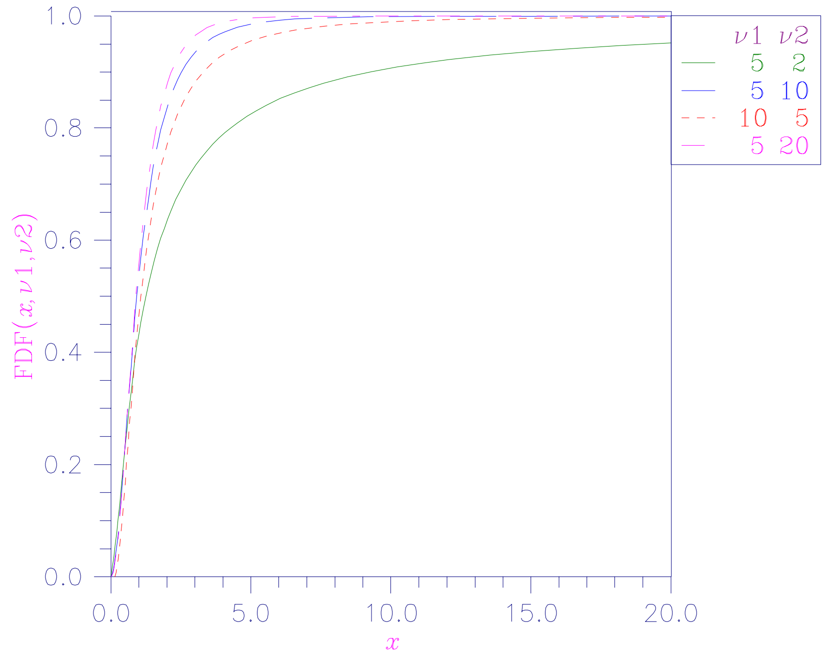


Figure 11-10 *F* Distribution Function

FIN

This function evaluates the inverse of the *F* distribution function.

Function Return Value

FIN — Function value. (Output)
The probability that an *F* random variable takes a value less than or equal to `FIN` is `P`.

Required Arguments

P — Probability for which the inverse of the *F* distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

DFN — Numerator degrees of freedom. (Input)
`DFN` must be positive.

DFD — Denominator degrees of freedom. (Input)
`DFD` must be positive.

FORTRAN 90 Interface

Generic: `FIN(P, DFN, DFD)`

Specific: The specific interface names are `S_FIN` and `D_FIN`.

FORTRAN 77 Interface

Single: `FIN(P, DFN, DFD)`

Double: The double precision function name is `DFIN`.

Example

In this example, we find the 99-th percentage point for an *F* random variable with 1 and 7 degrees of freedom.

```
      USE FIN_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       DFD, DFN, F, P
!
      CALL UMACH (2, NOUT)
      P   = 0.99
      DFN = 1.0
      DFD = 7.0
      F   = FIN(P,DFN,DFD)
      WRITE (NOUT,99999) F
99999 FORMAT (' The F(1,7) 0.01 critical value is ', F6.3)
      END
```

Output

```
The F(1,7) 0.01 critical value is 12.246
```

Comments

Informational error

Type	Code	
4	4	`FIN` is set to machine infinity since overflow would occur upon modifying the inverse value for the F distribution with the result obtained from the inverse `BETA` distribution.

Description

Function `FIN` evaluates the inverse distribution function of a Snedecor's F random variable with `DFN` numerator degrees of freedom and `DFD` denominator degrees of freedom. The function is evaluated by making a transformation to a beta random variable and then using the routine `BETIN` (page 212). If X is an F variate with ν_1 and ν_2 degrees of freedom and $Y = \nu_1 X/(\nu_2 + \nu_1 X)$, then Y is a beta variate with parameters $p = \nu_1/2$ and $q = \nu_2/2$. If $P \le 0.5$, `FIN` uses this relationship directly; otherwise, it also uses a relationship between F random variables that can be expressed as follows, using routine `FDF` (page 222), which is the F cumulative distribution function:

```
FDF(F, DFN, DFD) = 1.0 - FDF(1.0/F, DFD, DFN)
```

GAMDF

This function evaluates the gamma distribution function.

Function Return Value

GAMDF — Function value, the probability that a gamma random variable takes a value less than or equal to `X`. (Output)

Required Arguments

X — Argument for which the gamma distribution function is to be evaluated. (Input)

A — The shape parameter of the gamma distribution. (Input)
This parameter must be positive.

FORTRAN 90 Interface

Generic: `GAMDF(X, A)`

Specific: The specific interface names are `S_GAMDF` and `D_GAMDF`.

FORTRAN 77 Interface

Single: `GAMDF(X, A)`

Double: The double precision function name is `DGAMDF`.

Example

Suppose `X` is a gamma random variable with a shape parameter of 4. (In this case, it has an *Erlang distribution* since the shape parameter is an integer.) In this example, we find the probability that `X` is less than 0.5 and the probability that `X` is between 0.5 and 1.0.

```
      USE GAMDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       A, P, X
!
      CALL UMACH (2, NOUT)
      A = 4.0
      X = 0.5
      P = GAMDF(X,A)
      WRITE (NOUT,99998) P
99998 FORMAT (' The probability that X is less than 0.5 is ', F6.4)
      X = 1.0
      P = GAMDF(X,A) - P
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is between 0.5 and 1.0 is ', &
            F6.4)
      END
```

Output

```
The probability that X is less than 0.5 is 0.0018
The probability that X is between 0.5 and 1.0 is 0.0172
```

Comments

Informational error

Type	Code	
1	2	Since the input argument `X` is less than zero, the distribution function is set to zero.

Description

Function `GAMDF` evaluates the distribution function, F, of a gamma random variable with shape parameter a; that is,

$$F(x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt$$

where $\Gamma(\cdot)$ is the gamma function. (The gamma function is the integral from 0 to ∞ of the same integrand as above). The value of the distribution function at the point x is the probability that the random variable takes a value less than or equal to x.

The gamma distribution is often defined as a two-parameter distribution with a scale parameter b (which must be positive), or even as a three-parameter distribution in which the third parameter c is a location parameter. In the most general case, the probability density function over (c, ∞) is

$$f(t) = \frac{1}{b^a \Gamma(a)} e^{-(t-c)/b} (x-c)^{a-1}$$

If T is such a random variable with parameters a, b, and c, the probability that $T \le t_0$ can be obtained from `GAMDF` by setting `X` = $(t_0 - c)/b$.

If `X` is less than a or if `X` is less than or equal to 1.0, `GAMDF` uses a series expansion. Otherwise, a continued fraction expansion is used. (See Abramowitz and Stegun, 1964.)

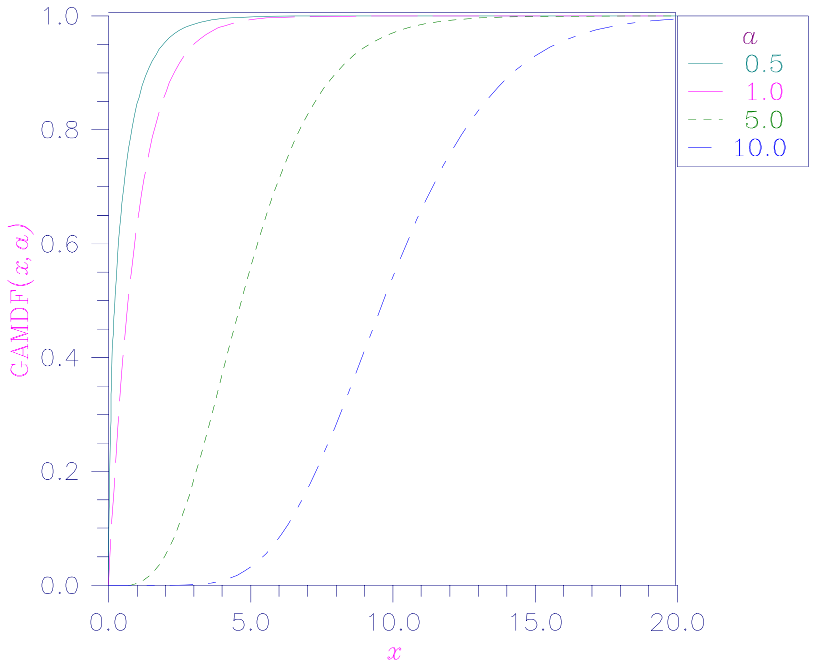


Figure 11-11 Gamma Distribution Function

TDF

This function evaluates the Student's t distribution function.

Function Return Value

TDF — Function value, the probability that a Student's t random variable takes a value less than or equal to the input `T`. (Output)

Required Arguments

T — Argument for which the Student's *t* distribution function is to be evaluated. (Input)

DF — Degrees of freedom. (Input)
DF must be greater than or equal to 1.0.

FORTRAN 90 Interface

Generic: TDF(T, DF)

Specific: The specific interface names are S_TDF and D_TDF.

FORTRAN 77 Interface

Single: TDF(T, DF)

Double: The double precision function name is DTDF.

Example

In this example, we find the probability that a *t* random variable with 6 degrees of freedom is greater in absolute value than 2.447. We use the fact that *t* is symmetric about 0.

```
      USE TDF_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       DF, P, T
!
      CALL UMACH (2, NOUT)
      T  = 2.447
      DF = 6.0
      P  = 2.0*TDF(-T,DF)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that a t(6) variate is greater ', &
            'than 2.447 in', /, ' absolute value is ', F6.4)
      END
```

Output

```
The probability that a t(6) variate is greater than 2.447 in
absolute value is 0.0500
```

Description

Function TDF evaluates the distribution function of a Student's *t* random variable with DF degrees of freedom. If the square of T is greater than or equal to DF, the relationship of a *t* to an *F* random variable (and subsequently, to a beta random variable) is exploited; and routine BETDF (page 209) is used. Otherwise, the method described by Hill (1970) is used. If DF is not an integer, if DF is greater than 19, or if DF is greater than 200, a Cornish-Fisher expansion is used to evaluate the distribution function. If DF is less than 20 and ABS(T) is less than 2.0, a trigonometric series (see Abramowitz and Stegun, 1964, equations 26.7.3 and 26.7.4, with some rearrangement) is used. For the remaining cases, a series given by Hill (1970) that converges well for large values of `T` is used.

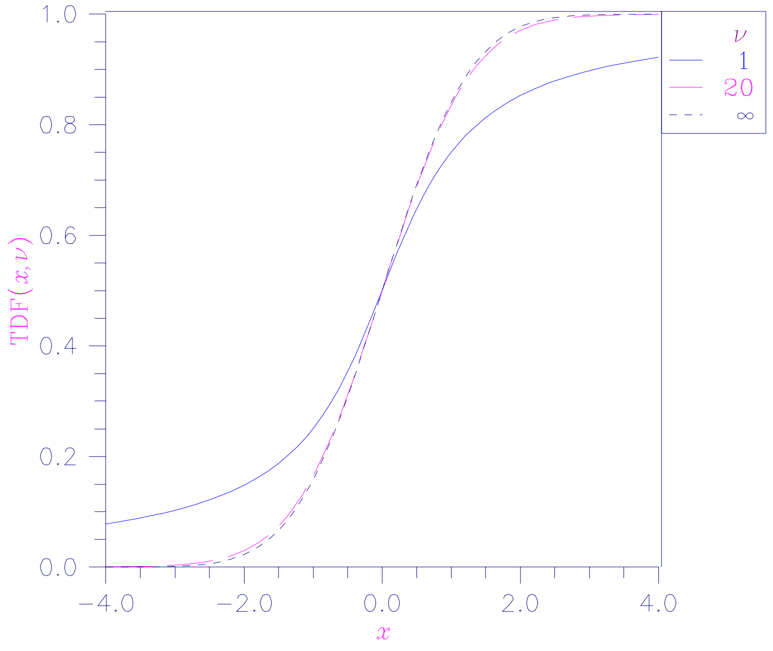


Figure 11-12 Student's *t* Distribution Function

TIN

This function evaluates the inverse of the Student's *t* distribution function.

Function Return Value

TIN — Function value. (Output)
The probability that a Student's *t* random variable takes a value less than or equal to `TIN` is `P`.

Required Arguments

P — Probability for which the inverse of the Student's *t* distribution function is to be evaluated. (Input)
`P` must be in the open interval (0.0, 1.0).

DF — Degrees of freedom. (Input)
`DF` must be greater than or equal to 1.0.

FORTRAN 90 Interface

Generic: `TIN(P, DF)`

Specific: The specific interface names are `S_TIN` and `D_TIN`.

FORTRAN 77 Interface

Single: `TIN(P, DF)`

Double: The double precision function name is `DTIN`.

Example

In this example, we find the 0.05 critical value for a two-sided *t* test with 6 degrees of freedom.

```
      USE TIN_INT
      USE UMACH_INT
      INTEGER    NOUT
      REAL       DF, P, T
!
      CALL UMACH (2, NOUT)
      P  = 0.975
      DF = 6.0
      T  = TIN(P,DF)
      WRITE (NOUT,99999) T
99999 FORMAT (' The two-sided t(6) 0.05 critical value is ', F6.3)
      END
```

Output

```
The two-sided t(6) 0.05 critical value is 2.447
```

Comments

Informational error

Type	Code	
4	3	`TIN` is set to machine infinity since overflow would occur upon modifying the inverse value for the *F* distribution with the result obtained from the inverse β distribution.

Description

Function `TIN` evaluates the inverse distribution function of a Student's *t* random variable with `DF` degrees of freedom. Let ν = `DF`. If ν equals 1 or 2, the inverse can be obtained in closed form; if ν is between 1 and 2, the relationship of a *t* to a beta random variable is exploited and routine `BETIN` (page 212) is used to evaluate the inverse; otherwise the algorithm of Hill (1970) is used. For small values of ν greater than 2, Hill's algorithm inverts an integrated expansion in $1/(1 + t^2/\nu)$ of the *t* density. For larger values, an asymptotic inverse Cornish-Fisher type expansion about normal deviates is used.

TNDF

This function evaluates the noncentral Student's *t* distribution function.

Function Return Value

TNDF — Function value, the probability that a noncentral Student's *t* random variable takes a value less than or equal to `T`. (Output)

Required Arguments

T — Argument for which the noncentral Student's *t* distribution function is to be evaluated. (Input)

IDF — Number of degrees of freedom of the noncentral Student's *t* distribution. (Input) `IDF` must be positive.

DELTA — The noncentrality parameter. (Input)

FORTRAN 90 Interface

Generic: `TNDF(T, IDF, DELTA)`

Specific: The specific interface names are `S_TNDF` and `D_TNDF`.

FORTRAN 77 Interface

Single: `TNDF(T, IDF, DELTA)`

Double: The double precision function name is `DTNDF`.

Example

Suppose *T* is a noncentral *t* random variable with 6 degrees of freedom and noncentrality parameter 6. In this example, we find the probability that *T* is less than 12.0. (This can be checked using the table on page 111 of Owen, 1962, with $\eta = 0.866$, which yields $\lambda = 1.664$.)

```
      USE TNDF_INT
      USE UMACH_INT
      INTEGER    IDF, NOUT
      REAL       DELTA, P, T
!
      CALL UMACH (2, NOUT)
      IDF   = 6
      DELTA = 6.0
      T     = 12.0
      P     = TNDF(T,IDF,DELTA)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that T is less than 12.0 is ', F6.4)
      END
```

Output

```
The probability that T is less than 12.0 is 0.9501
```

Description

Function `TNDF` evaluates the distribution function F of a noncentral t random variable with `IDF` degrees of freedom and noncentrality parameter `DELTA`; that is, with ν = `IDF`, δ = `DELTA` , and t_0 = `T`,

$$F(t_0) = \int_{-\infty}^{t_0} \frac{\nu^{\nu/2} e^{-\delta^2/2}}{\sqrt{\pi}\Gamma(\nu/2)(\nu + x^2)^{(\nu+1)/2}}$$

$$\sum_{i=0}^{\infty} \Gamma((\nu+i+1)/2)\left(\frac{\delta^i}{i!}\right)\left(\frac{2x^2}{\nu+x^2}\right)^{i/2} dx$$

where $\Gamma(\cdot)$ is the gamma function. The value of the distribution function at the point t_0 is the probability that the random variable takes a value less than or equal to t_0.

The noncentral t random variable can be defined by the distribution function above, or alternatively and equivalently, as the ratio of a normal random variable and an independent chi-squared random variable. If w has a normal distribution with mean δ and variance equal to one, u has an independent chi-squared distribution with ν degrees of freedom, and

$$\frac{w}{\sqrt{u/\nu}}$$

then x has a noncentral t distribution with ν degrees of freedom and noncentrality parameter δ.

The distribution function of the noncentral t can also be expressed as a double integral involving a normal density function (see, for example, Owen, 1962, page 108). The function `TNDF` uses the method of Owen (1962, 1965), which uses repeated integration by parts on that alternate expression for the distribution function.

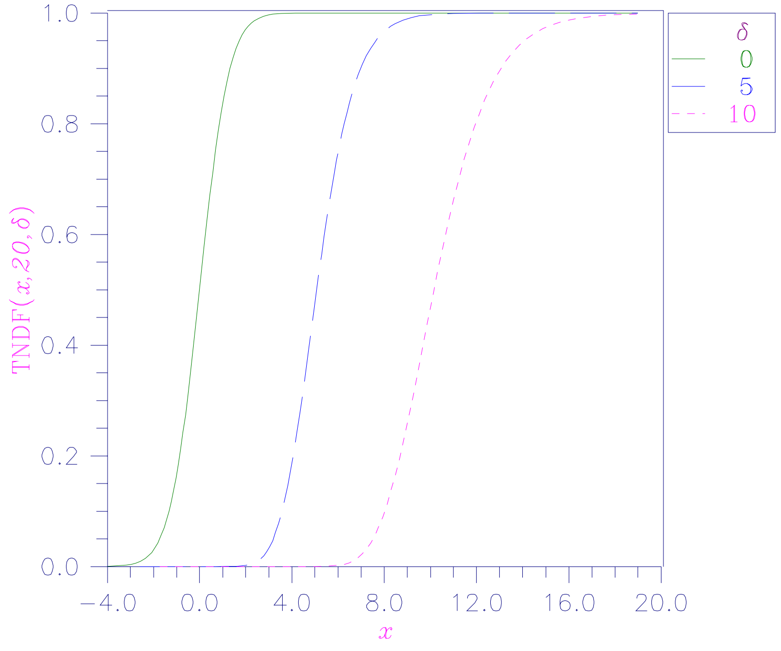


Figure 11-13 Noncentral Student's *t* Distribution Function

GCDF

This function evaluates a general continuous cumulative distribution function given ordinates of the density.

Function Return Value

GCDF — Function value, the probability that a random variable whose density is given in `F` takes a value less than or equal to `X0`. (Output)

Required Arguments

X0 — Point at which the distribution function is to be evaluated. (Input)

X — Array containing the abscissas or the endpoints. (Input)
If `IOPT` = 1 or 3, `X` is of length 2. If `IOPT` = 2 or 4, `X` is of length `M`. For `IOPT` = 1 or 3, `X`(1) contains the lower endpoint of the support of the distribution and `X`(2) is the upper endpoint. For `IOPT` = 2 or 4, `X` contains, in strictly increasing order, the abscissas such that `X`(`I`) corresponds to `F`(`I`).

F — Vector of length `M` containing the probability density ordinates corresponding to increasing abscissas. (Input)

If `IOPT` = 1 or 3; for `I` = 1, 2, …, `M`, `F(I)` corresponds to
`X`(1) + (`I` – 1) `*` (`X`(2) – `X`(1))/(`M` – 1); otherwise, `F` and `X` correspond one for one.

Optional Arguments

IOPT — Indicator of the method of interpolation. (Input)
Default: `IOPT` = 1.

IOPT	**Interpolation Method**
1	Linear interpolation with equally spaced abscissas.
2	Linear interpolation with possibly unequally spaced abscissas.
3	A cubic spline is fitted to equally spaced abscissas.
4	A cubic spline is fitted to possibly unequally spaced abscissas.

M — Number of ordinates of the density supplied. (Input)
`M` must be greater than 1 for linear interpolation (`IOPT` = 1 or 2) and greater than 3 if a curve is fitted through the ordinates (`IOPT` = 3 or 4).
Default: `M` = size (`F`,1).

FORTRAN 90 Interface

Generic: `GCDF (X0, X, F [,…])`

Specific: The specific interface names are `S_GCDF` and `D_GCDF`.

FORTRAN 77 Interface

Single: `GCDF(X0, IOPT, M, X, F)`

Double: The double precision functin name is `DGCDF`.

Example

In this example, we evaluate the beta distribution function at the point 0.6. The probability density function of a beta random variable with parameters *p* and *q* is

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1}(1-x)^{q-1} \quad \text{for } 0 \le x \le 1$$

where Γ(·) is the gamma function. The density is equal to 0 outside the interval [0, 1]. We compute a constant multiple (we can ignore the constant gamma functions) of the density at 300 equally spaced points and input this information in `X` and `F`. Knowing that the probability density of this distribution is very peaked in the vicinity of 0.5, we could perhaps get a better fit by using unequally spaced abscissas, but we will keep it simple. Note that this is the same example as one used in the description of routine `BETDF` (page 209). The result from `BETDF` would be expected to be more accurate than that from `GCDF` since `BETDF` is designed specifically for this distribution.

```
      USE GCDF_INT
      USE UMACH_INT
      INTEGER    M
      PARAMETER  (M=300)
!
      INTEGER    I, IOPT, NOUT
      REAL       F(M), H, P, PIN1, QIN1, X(2), X0, XI
!
      CALL UMACH (2, NOUT)
      X0   = 0.6
      IOPT = 3
!                                 Initializations for a beta(12,12)
!                                 distribution.
      PIN1 = 11.0
      QIN1 = 11.0
      XI   = 0.0
      H    = 1.0/(M-1.0)
      X(1) = XI
      F(1) = 0.0
      XI   = XI + H
!                                 Compute ordinates of the probability
!                                 density function.
      DO 10  I=2, M - 1
         F(I) = XI**PIN1*(1.0-XI)**QIN1
         XI   = XI + H
   10 CONTINUE
      X(2) = 1.0
      F(M) = 0.0
      P    = GCDF(X0,X,F, IOPT=IOPT)
      WRITE (NOUT,99999) P
99999 FORMAT (' The probability that X is less than 0.6 is ', F6.4)
      END
```

Output

```
The probability that X is less than 0.6 is 0.8364
```

Comments

Workspace may be explicitly provided, if desired, by the use of G4DF/DG4DF.
The reference is:

G4DF(P, IOPT, M, X, F, WK, IWK)

The arguments in addition to those of GCDF are:

WK — Work vector of length 5 * M if IOPT = 3, and of length 4 * M if IOPT = 4.

IWK — Work vector of length M.

Description

Function GCDF evaluates a continuous distribution function, given ordinates of the probability density function. It requires that the range of the distribution be specified in X. For distributions with infinite ranges, endpoints must be chosen so that most of the probability content is included. The function GCDF first fits a curve to the points given in X and F with either a piecewise linear interpolant or a C^1 cubic spline interpolant based on a method by Akima (1970). Function GCDF then determines the area, A, under the curve. (If the distribution were of finite range and if the fit were exact, this area would be 1.0.) Using the same fitted curve, GCDF next determines the area up to the point x_0 (= X0). The value returned is the area up to x_0 divided by A. Because of the scaling by A, it is not assumed that the integral of the density defined by X and F is 1.0.

For most distributions, it is likely that better approximations to the distribution function are obtained when IOPT equals 3 or 4, that is, when a cubic spline is used to approximate the function. It is also likely that better approximations can be obtained when the abscissas are chosen more densely over regions where the density and its derivatives (when they exist) are varying greatly.

GCIN

Evaluates the inverse of a general continuous cumulative distribution function given ordinates of the density.

Required Arguments

P — Probability for which the inverse of the distribution function is to be evaluated. (Input)
P must be in the open interval (0.0, 1.0).

X — Array containing the abscissas or the endpoints. (Input)
If IOPT = 1 or 3, X is of length 2. If IOPT = 2 or 4, X is of length M. For IOPT = 1 or 3, X(1) contains the lower endpoint of the support of the distribution and X(2) is the upper endpoint. For IOPT = 2 or 4, X contains, in strictly increasing order, the abscissas such that X(I) corresponds to F(I).

F — Vector of length M containing the probability density ordinates corresponding to increasing abscissas. (Input)
If IOPT = 1 or 3, for I = 1, 2, …, M, F(I) corresponds to
X(1) + (I − 1) * (X(2) − X(1))/(M − 1); otherwise, F and X correspond one for one.

GCIN — Function value. (Output)
The probability that a random variable whose density is given in F takes a value less than or equal to GCIN is P.

Optional Arguments

IOPT — Indicator of the method of interpolation. (Input)
Default: IOPT = 1.

IOPT	Interpolation Method
1	Linear interpolation with equally spaced abscissas.
2	Linear interpolation with possibly unequally spaced abscissas.

3	A cubic spline is fitted to equally spaced abscissas.
4	A cubic spline is fitted to possibly unequally spaced abscissas.

M — Number of ordinates of the density supplied. (Input)
`M` must be greater than 1 for linear interpolation (`IOPT` = 1 or 2) and greater than 3 if a curve is fitted through the ordinates (`IOPT` = 3 or 4).
Default: `M` = size (`F`,1).

FORTRAN 90 Interface

Generic: `CALL GCIN (P, X, F [,…])`

Specific: The specific interface names are `S_GCIN` and `D_GCIN`.

FORTRAN 77 Interface

Single: `CALL GCIN(P, IOPT, M, X, F)`

Double: The double precision function name is `DGCIN`.

Example

In this example, we find the 90-th percentage point for a beta random variable with parameters 12 and 12. The probability density function of a beta random variable with parameters *p* and *q* is

$$f(x) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} x^{p-1} (1-x)^{q-1} \quad \text{for } 0 \le x \le 1$$

where Γ(·) is the gamma function. The density is equal to 0 outside the interval [0, 1]. With *p* = *q*, this is a symmetric distribution. Knowing that the probability density of this distribution is very peaked in the vicinity of 0.5, we could perhaps get a better fit by using unequally spaced abscissas, but we will keep it simple and use 300 equally spaced points. Note that this is the same example that is used in the description of routine `BETIN` (page 212). The result from `BETIN` would be expected to be more accurate than that from `GCIN` since `BETIN` is designed specifically for this distribution.

```
      USE GCIN_INT
      USE UMACH_INT
      USE BETA_INT
      INTEGER    M
      PARAMETER  (M=300)
!
      INTEGER    I, IOPT, NOUT
      REAL       C, F(M), H, P, PIN, PIN1, QIN, QIN1, &
                X(2), X0, XI
!
      CALL UMACH (2, NOUT)
      P    = 0.9
      IOPT = 3
!                                Initializations for a beta(12,12)
!                                distribution.
```

```
      PIN  = 12.0
      QIN  = 12.0
      PIN1 = PIN - 1.0
      QIN1 = QIN - 1.0
      C    = 1.0/BETA(PIN,QIN)
      XI   = 0.0
      H    = 1.0/(M-1.0)
      X(1) = XI
      F(1) = 0.0
      XI   = XI + H
!                                  Compute ordinates of the probability
!                                  density function.
      DO 10  I=2, M - 1
         F(I) = C*XI**PIN1*(1.0-XI)**QIN1
         XI   = XI + H
   10 CONTINUE
      X(2) = 1.0
      F(M) = 0.0
      X0   = GCIN(P,X,F, IOPT=IOPT)
      WRITE (NOUT,99999) X0
99999 FORMAT (' X is less than ', F6.4, ' with probability 0.9.')
      END
```

Output

```
X is less than 0.6304 with probability 0.9.
```

Comments

Workspace may be explicitly provided, if desired, by the use of G3IN/DG3IN. The reference is

```
G3IN(P, IOPT, M, X, F, WK, IWK)
```

The arguments in addition to those of GCIN are:

WK — Work vector of length 5 * M if IOPT = 3, and of length 4 * M if IOPT = 4.

IWK — Work vector of length M.

Description

Function GCIN evaluates the inverse of a continuous distribution function, given ordinates of the probability density function. The range of the distribution must be specified in X. For distributions with infinite ranges, endpoints must be chosen so that most of the probability content is included.

The function GCIN first fits a curve to the points given in X and F with either a piecewise linear interpolant or a C^1 cubic spline interpolant based on a method by Akima (1970). Function GCIN then determines the area, A, under the curve. (If the distribution were of finite range and if the fit were exact, this area would be 1.0.) It next finds the maximum abscissa up to which the area is less than AP and the minimum abscissa up to which the area is greater than AP. The routine then interpolates for the point corresponding to AP. Because of the scaling by A, it is not assumed that the integral of the density defined by X and F is 1.0.

For most distributions, it is likely that better approximations to the distribution function are obtained when `IOPT` equals 3 or 4, that is, when a cubic spline is used to approximate the function. It is also likely that better approximations can be obtained when the abscissas are chosen more densely over regions where the density and its derivatives (when they exist) are varying greatly.

Chapter 12: Mathieu Functions

Routines

Usage Notes

Mathieu's equation is

$$\frac{d^2y}{dv^2} + \left(a - 2q\cos 2v\right)y = 0$$

It arises from the solution, by separation of variables, of Laplace's equation in elliptical coordinates, where a is the separation constant and q is related to the ellipticity of the coordinate system. If we let $t = \cos v$, then Mathieu's equation can be written as

$$\left(1 - t^2\right)\frac{d^2y}{dt^2} - t\frac{dy}{dt} + \left(a + 2q - 4qt^2\right)y = 0$$

For various physically important problems, the solution $y(t)$ must be periodic. There exist, for particular values of a, periodic solutions to Mathieu's equation of period $k\pi$ for any integer k. These particular values of a are called *eigenvalues* or *characteristic values*. They are computed using the routine MATEE (page 241).

There exist sequences of both even and odd periodic solutions to Mathieu's equation. The even solutions are computed by MATCE (page 244). The odd solutions are computed by MATSE (page 246).

MATEE

Evaluates the eigenvalues for the periodic Mathieu functions.

Required Arguments

Q — Parameter. (Input)

ISYM — Symmetry indicator. (Input)

ISYM	**Meaning**
0	Even
1	Odd

IPER — Periodicity indicator. (Input)

ISYM	**Period**
0	pi
1	2 * pi

EVAL — Vector of length N containing the eigenvalues. (Output)

Optional Arguments

N — Number of eigenvalues to be computed. (Input)
Default: N = size (EVAL,1)

FORTRAN 90 Interface

Generic: `CALL MATEE (Q, ISYM, IPER, EVAL [,…])`

Specific: The specific interface names are S_MATEE and D_MATEE.

FORTRAN 77 Interface

Single: `CALL MATEE (Q, N, ISYM, IPER, EVAL)`

Double: The double precision function name is DMATEE.

Example

In this example, the eigenvalues for $q = 5$, even symmetry, and π periodicity are computed and printed.

```
      USE UMACH_INT
      USE MATEE_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
!
      INTEGER    ISYM, IPER, K, NOUT
      REAL       Q, EVAL(N)
!                                 Compute
      Q    = 5.0
      ISYM = 0
      IPER = 0
      CALL MATEE (Q, ISYM, IPER, EVAL)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) 2*K-2, EVAL(K)
   10 CONTINUE
99999 FORMAT (' Eigenvalue', I2, ' = ', F9.4)
      END
```

Output

```
Eigenvalue 0 =   -5.8000
Eigenvalue 2 =    7.4491
Eigenvalue 4 =   17.0966
Eigenvalue 6 =   36.3609
Eigenvalue 8 =   64.1989
Eigenvalue10 =  100.1264
Eigenvalue12 =  144.0874
Eigenvalue14 =  196.0641
Eigenvalue16 =  256.0491
Eigenvalue18 =  324.0386
```

Comments

1. Workspace may be explicitly provided, if desired, by use of `M2TEE/DM2TEE`. The reference is

   `CALL M2TEE (Q, N, ISYM, IPER, EVAL, NORDER, WORKD, WORKE)`

   The additional arguments are as follows:

   NORDER — Order of the matrix whose eigenvalues are computed. (Input)

   WORKD — Work vector of size `NORDER`. (Input/Output)
   If `EVAL` is large enough then `EVAL` and `WORKD` can be the same vector.

   WORKE — Work vector of size `NORDER`. (Input/Output)

2. Informational error

Type	Code	
4	1	The iteration for the eigenvalues did not converge.

Description

The eigenvalues of Mathieu's equation are computed by a method due to Hodge (1972). The desired eigenvalues are the same as the eigenvalues of the following symmetric, tridiagonal matrix:

$$\begin{bmatrix} W_0 & qX_0 & 0 & 0 & \cdots \\ qX_0 & W_2 & qX_2 & 0 & \cdots \\ 0 & qX_2 & W_4 & qX_4 & \cdots \\ 0 & 0 & qX_4 & W_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Here,

$$X_m = \begin{cases} \sqrt{2} & \texttt{if ISYM} = \texttt{IPER} = m = 0 \\ 1 & \textbf{otherwise} \end{cases}$$

$$W_m = \left[m + \texttt{IPER} + 2\left(1 - \texttt{IPER}\right)\texttt{ISYM} \right]^2 + V_m$$

where

$$V_m = \begin{cases} +q & \text{if } \texttt{IPER} = 1, \texttt{ISYM} = 0 \text{ and } m = 0 \\ -q & \text{if } \texttt{IPER} = 1, \texttt{ISYM} = 1 \text{ and } m = 0 \\ 0 & \textbf{otherwise} \end{cases}$$

Since the above matrix is semi-infinite, it must be truncated before its eigenvalues can be computed. Routine `MATEE` computes an estimate of the number of terms needed to get accurate results. This estimate can be overridden by calling `M2TEE` with `NORDER` equal to the desired order of the truncated matrix.

The eigenvalues of this matrix are computed using the routine `EVLSB` found in the IMSL MATH/LIBRARY Chapter 2.

MATCE

Evaluates a sequence of even, periodic, integer order, real Mathieu functions.

Required Arguments

X — Argument for which the sequence of Mathieu functions is to be evaluated. (Input)

Q — Parameter. (Input)
The parameter `Q` must be positive.

N — Number of elements in the sequence. (Input)

CE — Vector of length `N` containing the values of the function through the series. (Output)
`CE(I)` contains the value of the Mathieu function of order `I` − 1 at `X` for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL MATCE (X, Q, N, CE)`

Specific: The specific interface names are `S_MATCE` and `D_MATCE`.

FORTRAN 77 Interface

Single: `CALL MATCE (X, Q, N, CE)`

Double: The double precision name is `DMATCE`.

Example 1

In this example, $ce_n(x = \pi/4, q = 1)$, $n = 0, \ldots, 9$ is computed and printed.

```
      USE CONST_INT
      USE MATCE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)

!
      INTEGER    K, NOUT
      REAL       CE(N), Q, X
!                                 Compute
      Q = 1.0
      X = CONST('PI')
      X = 0.25* X
      CALL MATCE (X, Q, N, CE)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, Q, CE(K)
   10 CONTINUE
99999 FORMAT (' ce sub', I2, ' (', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
ce sub 0 ( 0.785, 1.000) =  0.654
ce sub 1 ( 0.785, 1.000) =  0.794
ce sub 2 ( 0.785, 1.000) =  0.299
ce sub 3 ( 0.785, 1.000) = -0.555
ce sub 4 ( 0.785, 1.000) = -0.989
ce sub 5 ( 0.785, 1.000) = -0.776
ce sub 6 ( 0.785, 1.000) = -0.086
ce sub 7 ( 0.785, 1.000) =  0.654
ce sub 8 ( 0.785, 1.000) =  0.998
ce sub 9 ( 0.785, 1.000) =  0.746
```

Comments

1. Workspace may be explicitly provided, if desired, by use of M2TCE/DM2TCE. The reference is

   ```
   CALL M2TCE (X, Q, N, CE, NORDER, NEEDEV, EVAL0, EVAL1, COEF,
   WORK, BSJ)
   ```

   The additional arguments are as follows:

   NORDER — Order of the matrix used to compute the eigenvalues. (Input)
   It must be greater than N. Routine MATSE computes NORDER by the following call to M3TEE.

   CALL M3TEE(Q, N, NORDER)

NEEDEV — Logical variable, if `.TRUE.`, the eigenvalues must be computed. (Input)

EVAL0 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 0 and `IPER` = 0. (Input/Output)
If `NEEDEV` is `.TRUE.`, then `EVAL0` is computed by `M2TCE`; otherwise, it must be set as an input value.

EVAL1 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 0 and `IPER` = 1. (Input/Output)
If `NEEDEV` is `.TRUE.`, then `EVAL1` is computed by `M2TCE`; otherwise, it must be set as an input value.

COEF — Real work vector of length `NORDER` + 4.

WORK — Real work vector of length `NORDER` + 4.

BSJ — Real work vector of length 2 * `NORDER` – 2.

2. Informational error

Type	Code	
4	1	The iteration for the eigenvalues did not converge.

Description

The eigenvalues of Mathieu's equation are computed using `MATEE` (page 241). The function values are then computed using a sum of Bessel functions, see Gradshteyn and Ryzhik (1965), equation 8.661.

Additional Examples

Example 2

In this example, we compute $\mathrm{ce}_n(x, q)$ for various values of n and x and a fixed value of q. To avoid having to recompute the eigenvalues, which depend on q but not on x, we compute the eigenvalues once and pass in their value to `M2TCE`. The eigenvalues are computed using `MATEE` (page 241). The routine `M3TEE` is used to compute `NORDER` based on `Q` and `N`. The arrays `BSJ`, `COEF` and `WORK` are used as temporary storage in `M2TCE`.

```
      USE IMSL_LIBRARIES

!                                  Declare variables
      INTEGER    MAXORD, N, NX
      PARAMETER  (MAXORD=100, N=4, NX=5)
!
      INTEGER    ISYM, K, NORDER, NOUT
      REAL       BSJ(2*MAXORD-2), CE(N), COEF(MAXORD+4)
      REAL       EVAL0(MAXORD), EVAL1(MAXORD), PI, Q, WORK(MAXORD+4), X
!                                  Compute NORDER
      Q = 1.0
      CALL M3TEE (Q, N, NORDER)
```

```
!
      CALL UMACH (2, NOUT)
      WRITE (NOUT, 99997) NORDER
!                                Compute eigenvalues
      ISYM = 0
      CALL MATEE (Q, ISYM, 0, EVAL0)
      CALL MATEE (Q, ISYM, 1, EVAL1)
!
      PI = CONST('PI')
!                                Compute function values
      WRITE (NOUT, 99998)
      DO 10  K=0, NX
         X = (K*PI)/NX
         CALL M2TCE(X, Q, N, CE, NORDER, .FALSE., EVAL0, EVAL1, &
          COEF, WORK, BSJ)
         WRITE (NOUT,99999) X, CE(1), CE(2), CE(3), CE(4)
   10 CONTINUE
!
99997 FORMAT (' NORDER = ', I3)
99998 FORMAT (/, 28X, 'Order', /, 20X, '0', 7X, '1', 7X, &
          '2', 7X, '3')
99999 FORMAT (' ce(', F6.3, ') = ', 4F8.3)
      END
```

Output

```
NORDER =  23
                            Order
                   0       1       2       3
ce( 0.000) =    0.385   0.856   1.086   1.067
ce( 0.628) =    0.564   0.838   0.574  -0.131
ce( 1.257) =    0.926   0.425  -0.575  -0.820
ce( 1.885) =    0.926  -0.425  -0.575   0.820
ce( 2.513) =    0.564  -0.838   0.574   0.131
ce( 3.142) =    0.385  -0.856   1.086  -1.067
```

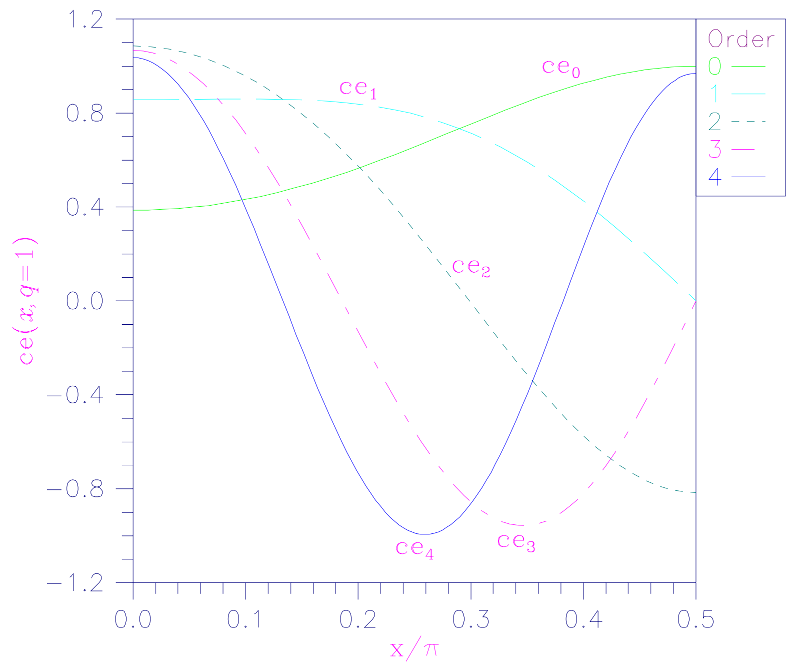


Figure 12-1 Plot of $\text{ce}_n(x, q = 1)$

MATSE

Evaluates a sequence of odd, periodic, integer order, real Mathieu functions.

Required Arguments

X — Argument for which the sequence of Mathieu functions is to be evaluated. (Input)

Q — Parameter. (Input)
The parameter `Q` must be positive.

N — Number of elements in the sequence. (Input)

SE — Vector of length `N` containing the values of the function through the series. (Output)
`SE(I)` contains the value of the Mathieu function of order `I` at `X` for `I` = 1 to `N`.

FORTRAN 90 Interface

Generic: `CALL MATSE (X, Q, N, SE)`

Specific: The specific interface names are `S_MATSE` and `D_MATSE`.

FORTRAN 77 Interface

Single: `CALL MATSE (X, Q, N, SE)`

Double: The double precision function name is `DMATSE`.

Example

In this example, $\mathrm{se}_n(x = \pi/4, q = 10)$, $n = 0, \ldots, 9$ is computed and printed.

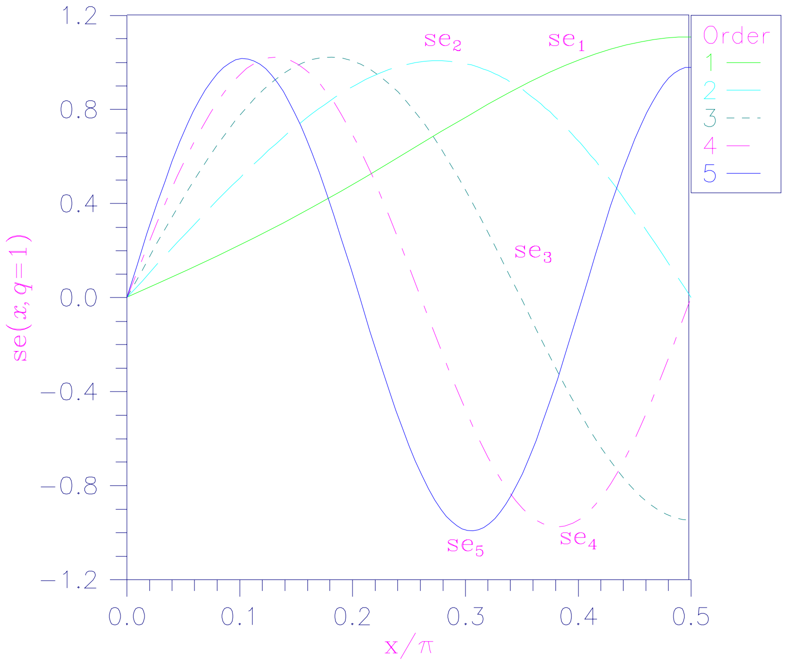


Figure 12-2 Plot of $\mathrm{se}_n(x, q = 1)$

```
      USE CONST_INT
      USE MATSE_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    N
      PARAMETER  (N=10)
!
      INTEGER    K, NOUT
      REAL       SE(N), Q, X
!                                 Compute
      Q = 10.0
      X = CONST('PI')
      X = 0.25* X
      CALL MATSE (X, Q, N, SE)
!                                 Print the results
      CALL UMACH (2, NOUT)
      DO 10  K=1, N
         WRITE (NOUT,99999) K-1, X, Q, SE(K)
```

```
   10 CONTINUE
99999 FORMAT (' se sub', I2, ' (', F6.3, ',', F6.3, ') = ', F6.3)
      END
```

Output

```
se sub 0 ( 0.785,10.000) =  0.250
se sub 1 ( 0.785,10.000) =  0.692
se sub 2 ( 0.785,10.000) =  1.082
se sub 3 ( 0.785,10.000) =  0.960
se sub 4 ( 0.785,10.000) =  0.230
se sub 5 ( 0.785,10.000) = -0.634
se sub 6 ( 0.785,10.000) = -0.981
se sub 7 ( 0.785,10.000) = -0.588
se sub 8 ( 0.785,10.000) =  0.219
se sub 9 ( 0.785,10.000) =  0.871
```

Comments

1. Workspace may be explicitly provided, if desired, by use of `M2TSE/DM2TSE`. The reference is

   ```
   CALL M2TSE (X, Q, N, SE, NORDER, NEEDEV, EVAL0,
   EVAL1, COEF, WORK, BSJ)
   ```

   The additional arguments are as follows:

   NORDER — Order of the matrix used to compute the eigenvalues. (Input)
   It must be greater than `N`. Routine `MATSE` computes `NORDER` by the following call to M3TEE.

   `CALL M3TEE (Q, N, NORDER)`

   NEEDEV — Logical variable, if `.TRUE.`, the eigenvalues must be computed. (Input)

   EVAL0 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 1 and `IPER` = 0. (Input/Output)
   If `NEEDEV` is `.TRUE.`, then `EVAL0` is computed by `M2TSE`; otherwise, it must be set as an input value.

   EVAL1 — Real work vector of length `NORDER` containing the eigenvalues computed by `MATEE` with `ISYM` = 1 and `IPER` = 1. (Input/Output)
   If `NEEDEV` is `.TRUE.`, then `EVAL1` is computed by `M2TSE`; otherwise, it must be set as an input value.

   COEF — Real work vector of length `NORDER` + 4.

   WORK — Real work vector of length `NORDER` + 4.

   BSI — Real work vector of length 2 * `NORDER` + 1.

2. Informational error

Type	Code	
4	1	The iteration for the eigenvalues did not converge.

Description

The eigenvalues of Mathieu's equation are computed using `MATEE` (page 241). The function values are then computed using a sum of Bessel functions, see Gradshteyn and Ryzhik (1965), equation 8.661.

Chapter 13: Miscellaneous Functions

Routines

Usage Notes

Many functions of one variable can be numerically computed using a Chebyshev series,

$$f(x) = \sum_{n=0}^{\infty} A_n T_n(x) \quad -1 \le x \le 1$$

A Chebyshev series is better for numerical computation than a Taylor series since the Chebyshev polynomials, $T_n(x)$, are better behaved than the monomials, x^n.

A Taylor series can be converted into a Chebyshev series using an algorithm of Fields and Wimp, (see Luke (1969), page 292).

Let

$$f(x) = \sum_{n=0}^{\infty} \xi_n x^n$$

be a Taylor series expansion valid for $|x| < 1$. Define

$$A_n = \frac{2}{4^n} \sum_{k=0}^{\infty} \frac{\left(n+\frac{1}{2}\right)_k (n+1)_k \, \xi_{n+k}}{(2n+1)_k \, k!}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol.

(Note that $(a)_{k+1} = (a + k)(a)_k$). Then,

$$f(x) = \tfrac{1}{2}T_0^*(x) + \sum\nolimits_{n=1}^{\infty} A_n T_n^*(x) \quad 0 \le x \le 1$$

where

$$T_n^*(x)$$

are the shifted Chebyshev polynomials,

$$T_n^*(x) = T_n^*(2x-1)$$

In an actual implementation of this algorithm, the number of terms in the Taylor series and the number of terms in the Chebyshev series must both be finite. If the Taylor series is an alternating series, then the error in using only the first M terms is less than $|\xi_{M+1}|$. The error in truncating the Chebyshev series to N terms is no more than

$$\sum\nolimits_{n=N+1}^{\infty} |c_n|$$

If the Taylor series is valid on $|x| < R$, then we can write

$$f(x) = \sum\nolimits_{n=0}^{\infty} \xi_n R^n (x/R)^n$$

and use $\xi_n R^n$ instead of ξ_n in the algorithm to obtain a Chebyshev series in x/R valid for $0 < x < R$. Unfortunately, if R is large, then the Chebyshev series converges more slowly.

The Taylor series centered at zero can be shifted to a Taylor series centered at c. Let $t = x - c$, so

$$f(x) = f(t+c) = \sum\nolimits_{n=0}^{\infty} \xi_n (t+c)^n = \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{j=0}^{n} \xi_n \binom{n}{j} c^{n-j} t^j$$
$$= \sum\nolimits_{n=0}^{\infty} \hat{\xi}_n t^n = \sum\nolimits_{n=0}^{\infty} \hat{\xi}_n (x-c)^n$$

By interchanging the order of the double sum, it can easily be shown that

$$\hat{\xi}_j = \sum\nolimits_{n=j}^{\infty} \binom{n}{j} c^{n-j} \xi_n$$

By combining scaling and shifting, we can obtain a Chebyshev series valid over any interval $[a, b]$ for which the original Taylor series converges.

The algorithm can also be applied to asymptotic series,

$$f(x) \sim \sum\nolimits_{n=0}^{\infty} \xi_n x^{-n} \text{ as } |x| \to \infty$$

by treating the series truncated to M terms as a polynomial in $1/x$. The asymptotic series is usually divergent; but if it is alternating, the error in truncating the series to M terms is less than $|\xi_{M+1}|/R^{M+1}$ for $R \le x < \infty$. Normally, as M increases, the error initially decreases to a small value and then increases without a bound. Therefore, there is a limit to the accuracy that can be obtained by increasing M. More accuracy can be obtained by increasing R. The optimal value of M depends on both the sequence ξ_j and R. For R fixed, the optimal value of M can be found by finding the value of M at which $|\xi_M|/R^M$ starts to increase.

Since we want a routine accurate to near machine precision, the algorithm must be implemented using somewhat higher precision than is normally used. This is best done using a symbolic computation package.

SPENC

This function evaluates a form of Spence's integral.

Function Return Value

SPENC — Function value. (Output)

Required Arguments

X — Argument for which the function value is desired. (Input)

FORTRAN 90 Interface

Generic: `SPENC (X)`

Specific: The specific interface names are `S_SPENC` and `D_SPENC`.

FORTRAN 77 Interface

Single: `SPENC (X)`

Double: The double precision function name is `DSPENC`.

Example

In this example, *s*(0.2) is computed and printed.

```
      USE SPENC_INT
      USE UMACH_INT
!                                 Declare variables
      INTEGER    NOUT
      REAL       VALUE, X
!                                 Compute
      X     = 0.2
      VALUE = SPENC(X)
!                                 Print the results
      CALL UMACH (2, NOUT)
      WRITE (NOUT,99999) X, VALUE
99999 FORMAT (' SPENC(', F6.3, ') = ', F6.3)
      END
```

Output

```
SPENC( 0.200) = 0.211
```

Description

The Spence dilogarithm function, $s(x)$, is defined to be

$$s(x) = -\int_0^x \frac{\ln|1-y|}{y}\, dy$$

For $|x| \le 1$, the uniformly convergent expansion

$$s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

is valid.

Spence's function can be used to evaluate much more general integral forms. For example,

$$c\int_0^z \frac{\log(ax+b)}{cx+d}dx = \log\left|\frac{a(cz+d)}{ad-bc}\right| - s\left(\frac{a(cz+d)}{ad-bc}\right)$$

INITS

This function Initializes the orthogonal series so the function value is the number of terms needed to insure the error is no larger than the requested accuracy.

Function Return Value

INITS — Number of terms needed to insure the error is no larger than `ETA`. (Output)

Required Arguments

OS — Vector of length `NOS` containing coefficients in an orthogonal series. (Input)

NOS — Number of coefficients in `OS`. (Input)

ETA — Requested accuracy of the series. (Input)
Contrary to the usual convention, `ETA` is a `REAL` argument to `INITDS`.

FORTRAN 90 Interface

Generic: `INITS(OS, NOS, ETA)`

Specific: The specific interface names are `INITS` and `INITDS`.

FORTRAN 77 Interface

Single: `INITS(OS, NOS, ETA)`

Double: The double precision function name is `INITDS`.

Comments

`ETA` will usually be chosen to be one tenth of machine precision.

Description

Function `INITS` initializes a Chebyshev series. The function `INITS` returns the number of terms in the series *s* of length *n* needed to insure that the error of the evaluated series is everywhere less than `ETA`. The number of input terms n must be greater than 1, so that a series of at least one term and an error estimate can be obtained. In addition, `ETA` should be larger than the absolute value of the last coefficient. If it is not, then all the terms of the series must be used, and no error estimate is available.

CSEVL

This function evaluates the *N*-term Chebyshev series.

Function Return Value

CSEVL — Function value. (Output)

Required Arguments

X — Argument at which the series is to be evaluated. (Input)

CS — Vector of length `N` containing the terms of a Chebyshev series. (Input)
In evaluating `CS`, only half of the first coefficient is summed.

Optional Arguments

N — Number of terms in the vector `CS`. (Input)|
Default: `N` = size(`CS`, 1)

FORTRAN 90 Interface

Generic: `CSEVL(X, CS [,…])`

Specific: The specific interface names are `S_CSEVL` and `D_CSEVL`.

FORTRAN 77 Interface

Single: `CSEVL(X, CS, N)`

Double: The double precision function name is `DCSEVL`.

Comments

Informational error

Type	Code	
3	7	`X` is outside the interval (−1.1, +1.1)

Description

Function `CSEVL` evaluates a Chebyshev series whose coefficients are stored in the array *s* of length *n* at the point *x*. The argument *x* must lie in the interval [−1, +1]. Other finite intervals can be linearly transformed to this canonical interval. Also, the number of terms in the series must be greater than zero but less than 1000. This latter limit is purely arbitrary; it is imposed in order to guard against the possibility of a floating point number being passed as an argument for *n*.

Reference Material

User Errors

IMSL routines attempt to detect user errors and handle them in a way that provides as much information to the user as possible. To do this, we recognize various levels of severity of errors, and we also consider the extent of the error in the context of the purpose of the routine; a trivial error in one situation may be serious in another. IMSL routines attempt to report as many errors as they can reasonably detect. Multiple errors present a difficult problem in error detection because input is interpreted in an uncertain context after the first error is detected.

What Determines Error Severity

In some cases, the user's input may be mathematically correct, but because of limitations of the computer arithmetic and of the algorithm used, it is not possible to compute an answer accurately. In this case, the assessed degree of accuracy determines the severity of the error. In cases where the routine computes several output quantities, if some are not computable but most are, an error condition exists. The severity depends on an assessment of the overall impact of the error.

Terminal errors

If the user's input is regarded as meaningless, such as `N` = −1 when "`N`" is the number of equations, the routine prints a message giving the value of the erroneous input argument(s) and the reason for the erroneous input. The routine will then cause the user's program to stop. An error in which the user's input is meaningless is the most severe error and is called a *terminal error*. Multiple terminal error messages may be printed from a single routine.

Informational errors

In many cases, the best way to respond to an error condition is simply to correct the input and rerun the program. In other cases, the user may want to take actions in the program itself based on errors that occur. An error that may be used as the basis for corrective action within the program is called an *informational error*. If an informational error occurs, a user-retrievable code is set. A routine can return at most one informational error for a single reference to the routine. The codes for the informational error codes are printed in the error messages.

Other errors

In addition to informational errors, IMSL routines issue error messages for which no user-retrievable code is set. Multiple error messages for this kind of error may be printed. These errors, which generally are not described in the documentation, include terminal errors as well as less serious errors. Corrective action within the calling program is not possible for these errors.

Kinds of Errors and Default Actions

Five levels of severity of errors are defined in the MATH/LIBRARY Special Functions. Each level has an associated PRINT attribute and a STOP attribute. These attributes have default settings (YES or NO), but they may also be set by the user. The purpose of having multiple error severity levels is to provide independent control of actions to be taken for errors of different severity. Upon return from an IMSL routine, exactly one error state exists. (A code 0 “error” is no informational error.) Even if more than one informational error occurs, only one message is printed (if the PRINT attribute is YES). Multiple errors for which no corrective action within the calling program is reasonable or necessary result in the printing of multiple messages (if the PRINT attribute for their severity level is YES). Errors of any of the severity levels except level 5 may be informational errors.

Level 1: Note. A *note* is issued to indicate the possibility of a trivial error or simply to provide information about the computations. Default attributes: PRINT=NO, STOP=NO

Level 2: Alert. An *alert* indicates that the user should be advised about events occurring in the software. Default attributes: PRINT=NO, STOP=NO

Level 3: Warning. A *warning* indicates the existence of a condition that may require corrective action by the user or calling routine. A warning error may be issued because the results are accurate to only a few decimal places, because some of the output may be erroneous but most of the output is correct, or because some assumptions underlying the analysis technique are violated. Often no corrective action is necessary and the condition can be ignored. Default attributes: PRINT=YES, STOP=NO

Level 4: Fatal. A *fatal* error indicates the existence of a condition that may be serious. In most cases, the user or calling routine must take corrective action to recover. Default attributes: PRINT=YES, STOP=YES

Level 5: Terminal. A *terminal* error is serious. It usually is the result of an incorrect specification, such as specifying a negative number as the number of equations. These errors may also be caused by various programming errors impossible to diagnose correctly in FORTRAN. The resulting error message may be perplexing to the user. In such cases, the user is advised to compare carefully the actual arguments passed to the routine with the dummy argument descriptions given in the documentation. Special attention should be given to checking argument order and data types.

A terminal error is not an informational error because corrective action within the program is generally not reasonable. In normal usage, execution is terminated immediately when a terminal error occurs. Messages relating to more than one terminal error are printed if they occur. Default attributes: PRINT=YES, STOP=YES

The user can set PRINT and STOP attributes by calling `ERSET` as described in “Routines for Error Handling.”

Errors in Lower-Level Routines

It is possible that a user’s program may call an IMSL routine that in turn calls a nested sequence of lower-level IMSL routines. If an error occurs at a lower level in such a nest of routines and if the lower-level routine cannot pass the information up to the original user-called routine, then a traceback of the routines is produced. The only common situation in which this can occur is when an IMSL routine calls a user-supplied routine that in turn calls another IMSL routine.

Routines for Error Handling

There are three ways in which the user may interact with the IMSL error handling system: (1) to change the default actions, (2) to retrieve the integer code of an informational error so as to take corrective action, and (3) to determine the severity level of an error. The routines to use are `ERSET`, `IERCD`, and `N1RTY`, respectively.

ERSET

Change the default printing or stopping actions when errors of a particular error severity level occur.

Required Arguments

IERSVR — Error severity level indicator. (Input)
If `IERSVR` = 0, actions are set for levels 1 to 5. If `IERSVR` is 1 to 5, actions are set for errors of the specified severity level.

IPACT — Printing action. (Input)

IPACT	**Action**
−1	Do not change current setting(s).
0	Do not print.
1	Print.
2	Restore the default setting(s).

ISACT — Stopping action. (Input)

ISACT	**Action**
−1	Do not change current setting(s).

0	Do not stop.
1	Stop.
2	Restore the default setting(s).

FORTRAN 90 Interface

Generic: `CALL ERSET (IERSVR, IPACT, ISACT)`

Specific: The specific interface name is `ERSET`.

FORTRAN 77 Interface

Single: `CALL ERSET (IERSVR, IPACT, ISACT)`

IERCD and N1RTY

The last two routines for interacting with the error handling system, `IERCD` and `N1RTY`, are `INTEGER` functions and are described in the following material.

`IERCD` retrieves the integer code for an informational error. Since it has no arguments, it may be used in the following way:

```
ICODE = IERCD( )
```

The function retrieves the code set by the most recently called IMSL routine.

`N1RTY` retrieves the error type set by the most recently called IMSL routine. It is used in the following way:

```
ITYPE = N1RTY(1)
```

`ITYPE` = 1, 2, 4, and 5 correspond to error severity levels 1, 2, 4, and 5, respectively. `ITYPE` = 3 and `ITYPE` = 6 are both warning errors, error severity level 3. While `ITYPE` = 3 errors are informational errors (`IERCD`( ) ≠ 0), `ITYPE` = 6 errors are not informational errors (`IERCD`( ) = 0).

For software developers requiring additional interaction with the IMSL error handling system, see Aird and Howell (1991).

Examples

Changes to Default Actions

Some possible changes to the default actions are illustrated below. The default actions remain in effect for the kinds of errors not included in the call to `ERSET`.

To turn off printing of warning error messages:
```
CALL ERSET (3, 0, −1)
```

To stop if warning errors occur:
```
CALL ERSET (3, −1, 1)
```

To print all error messages:

```
CALL ERSET (0, 1, −1)
```

To restore all default settings:

```
CALL ERSET (0, 2, 2)
```

Machine-Dependent Constants

The function subprograms in this section return machine-dependent information and can be used to enhance portability of programs between different computers. The routines `IMACH`, and `AMACH` describe the computer's arithmetic. The routine `UMACH` describes the input, ouput, and error output unit numbers.

IMACH

This function retrieves machine integer constants that define the arithmetic used by the computer.

Function Return Value

`IMACH`(1) = Number of bits per integer storage unit.

`IMACH`(2) = Number of characters per integer storage unit:

Integers are represented in M-digit, base A form as

$$\sigma \sum_{k=0}^{M} x_k A^k$$

where σ is the sign and $0 \le x_k < A$, $k = 0, \ldots, M$.

Then,

`IMACH`(3) = A, the base.

`IMACH`(4) = M, the number of base-A digits.

`IMACH`(5) = $A^M - 1$, the largest integer.

The machine model assumes that floating-point numbers are represented in normalized N-digit, base B form as

$$\sigma B^E \sum_{k=1}^{N} x_k B^{-k}$$

where σ is the sign, $0 < x_1 < B$, $0 \le x_k < B$, k = 2, …, N and $E_{\min} \le E \le E_{\max}$. Then, IMACH(6) = B, the base.

IMACH(7) = N_s, the number of base-B digits in single precision.

IMACH(8) = $E_{\min_s}$, the smallest single precision exponent.

IMACH(9) = $E_{\max_s}$, the largest single precision exponent.

IMACH(10) = N_d, the number of base-B digits in double precision.

IMACH(11) = $E_{\min_d}$, the smallest double precision exponent.

IMACH(12) = $E_{\max_d}$, the number of base-B digits in double precision

Required Arguments

I — Index of the desired constant. (Input)

FORTRAN 90 Interface

Generic: `IMACH (I)`

Specific: The specific interface name is `IMACH`.

FORTRAN 77 Interface

Single: `IMACH (I)`

AMACH

The function subprogram `AMACH` retrieves machine constants that define the computer's single-precision or double precision arithmetic. Such floating-point numbers are represented in normalized N-digit, base B form as

$$\sigma B^E \sum_{k=1}^{N} x_k B^{-k}$$

where σ is the sign, $0 < x_1 < B$, $0 \le x_k < B$, $k = 2, \ldots, N$ and

$$E_{\min} \le E \le E_{\max}$$

Function Return Value

AMACH(1) = $B^{E_{\min}-1}$, the smallest normalized positive number.

AMACH(2)=$B^{E_{\max}}\left(1-B^{-N}\right)$, the largest number.

AMACH(3)=B^{-N}, the smallest relative spacing.

AMACH(4)=B^{1-N}, the largest relative spacing.

AMACH(5) = $\log_{10}(B)$.

AMACH(6) = NaN (non-signaling not a number).

AMACH(7)=positive machine infinity.

AMACH(8)= negative machine infinity.

See Comment 1 for a description of the use of the generic version of this function.

See Comment 2 for a description of `min`, `max`, and `N`.

Required Arguments

I — Index of the desired constant. (Input)

FORTRAN 90 Interface

Generic: `AMACH (I)`

Specific: The specific interface names are `S_AMACH` and `D_AMACH`.

FORTRAN 77 Interface

Single: `AMACH (I)`

Double: The double precision name is `DMACH`.

Comments

1. If the generic version of this function is used, the immediate result must be stored in a variable before use in an expression. For example:

   ```
   X = AMACH(I)
   Y = SQRT(X)
   ```

   must be used rather than

   ```
   Y = SQRT(AMACH(I)).
   ```

   If this is too much of a restriction on the programmer, then the specific name can be used without this restriction..

2. Note that for single precision *B* = `IMACH`(6), *N*= `IMACH`(7).
   E`min` = `IMACH`(8), and *E*`max` = `IMACH`(9).
   For double precision *B* = `IMACH`(6), *N*= `IMACH`(10).
   E`min` = `IMACH`(11), and *E*`max` = `IMACH`(12).

3. The IEEE standard for binary arithmetic (see IEEE 1985) specifies *quiet* NaN (not a number) as the result of various invalid or ambiguous operations, such as 0/0. The intent is that `AMACH`(6) return a *quiet* NaN. On IEEE format computers that do not support a quiet NaN, a special value near `AMACH`(2) is returned for `AMACH`(6). On computers that do not have a special representation for infinity, `AMACH`(7) returns the same value as `AMACH`(2).

DMACH

See `AMACH`.

IFNAN(X)

This logical function checks if the argument `X` is NaN (not a number).

Function Return Value

***IFNAN* -** Logical function value. True is returned if the input argument is a `NAN`**.** Otherwise, False is returned. (Output)

Required Arguments

X – Argument for which the test for `NAN` is desired. (Input)

FORTRAN 90 Interface

Generic: `IFNAN(X)`

Specific: The specific interface names are `S_IFNAN` and `D_IFNAN`.

FORTRAN 77 Interface

Single: `IFNAN (X)`

Double: The double precision name is `DIFNAN`.

Example

```
      USE IFNAN_INT
      USE AMACH_INT
      USE UMACH_INT
      INTEGER      NOUT
      REAL         X
!
      CALL UMACH (2, NOUT)
!
      X = AMACH(6)
      IF (IFNAN(X)) THEN
         WRITE (NOUT,*) ' X is NaN (not a number).'
      ELSE
         WRITE (NOUT,*) ' X = ', X
      END IF
!
      END
```

Output

```
X is NaN (not a number).
```

Description

The logical function `IFNAN` checks if the single or double precision argument `X` is `NAN` (not a number). The function `IFNAN` is provided to facilitate the transfer of programs across computer systems. This is because the check for NaN can be tricky and not portable across computer systems that do not adhere to the IEEE standard. For example, on computers that support the IEEE standard for binary arithmetic (see IEEE 1985), NaN is specified as a bit format not equal to itself. Thus, the check is performed as

```
IFNAN = X .NE. X
```

On other computers that do not use IEEE floating-point format, the check can be performed as:

```
IFNAN = X .EQ. AMACH(6)
```

The function `IFNAN` is equivalent to the specification of the function Isnan listed in the Appendix, (IEEE 1985). The above example illustrates the use of `IFNAN`. If `X` is NaN, a message is printed instead of X. (Routine `UMACH`, which is described in the following section, is used to retrieve the output unit number for printing the message.)

UMACH

Routine `UMACH` sets or retrieves the input, output, or error output device unit numbers.

Required Arguments

N — Integer value indicating the action desired. If the value of `N` is negative, the input, output, or error output unit number is reset to `NUNIT`. If the value of `N` is positive, the input, output, or error output unit number is returned in `NUNIT`. See the table in argument `NUNIT` for legal values of `N`. (Input)

NUNIT — The unit number that is either retrieved or set, depending on the value of input argument `N`. (Input/Output)

The arguments are summarized by the following table:

N	**Effect**
1	Retrieves input unit number in `NUNIT`.
2	Retrieves output unit number in `NUNIT`.

N	Effect
3	Retrieves error output unit number in NUNIT.
−1	Sets the input unit number to NUNIT.
−2	Sets the output unit number to NUNIT.
−3	Sets the error output unit number to NUNIT.

FORTRAN 90 Interface

Generic: CALL UMACH (N, NUNIT)

Specific: The specific interface name is UMACH.

FORTRAN 77 Interface

Single: CALL UMACH (N, NUNIT)

Example

In the following example, a terminal error is issued from the MATH/LIBRARY AMACH function since the argument is invalid. With a call to UMACH, the error message will be written to a local file named “CHECKERR”.

```
      USE AMACH_INT
      USE UMACH_INT
      INTEGER     N, NUNIT
      REAL        X
!                                  Set Parameter
      N = 0
!
      NUNIT = 9
      CALL UMACH (-3, NUNIT)
      OPEN (UNIT=9,FILE='CHECKERR')
      X = AMACH(N)
      END
```

Output

The output from this example, written to “CHECKERR” is:

```
*** TERMINAL ERROR 5 from AMACH.  The argument must be between 1 and 8
***           inclusive. N = 0
```

Description

Routine UMACH sets or retrieves the input, output, or error output device unit numbers. UMACH is set automatically so that the default FORTRAN unit numbers for standard input, standard output, and standard error are used. These unit numbers can be changed by inserting a call to UMACH at the beginning of the main program that calls MATH/LIBRARY routines. If these unit numbers are changed from the standard values, the user should insert an appropriate `OPEN` statement in the calling program.

Reserved Names

When writing programs accessing IMSL MATH/LIBRARY Special Functions, the user should choose FORTRAN names that do not conflict with names of IMSL subroutines, functions, or named common blocks, such as the workspace common block `WORKSP` (page 261). The user needs to be aware of two types of name conflicts that can arise. The first type of name conflict occurs when a name (technically a *symbolic name*) is not uniquely defined within a program unit (either a main program or a subprogram). For example, such a name conflict exists when the name `BSJS` is used to refer both to a type `REAL` variable and to the IMSL routine `BSJS` in a single program unit. Such errors are detected during compilation and are easy to correct. The second type of name conflict, which can be more serious, occurs when names of program units and named common blocks are not unique. For example, such a name conflict would be caused by the user defining a routine named `WORKSP` and also referencing a MATH/LIBRARY Special Functions routine that uses the named common block `WORKSP`. Likewise, the user must not define a subprogram with the same name as a subprogram in MATH/LIBRARY Special Functions, that is referenced directly by the user's program or is referenced indirectly by other MATH/LIBRARY Special Functions subprograms.

MATH/LIBRARY Special Functions consists of many routines, some that are described in the *User's Manual* and others that are not intended to be called by the user and, hence, that are not documented. If the choice of names were completely random over the set of valid FORTRAN names and if a program uses only a small subset of MATH/LIBRARY Special Functions, the probability of name conflicts is very small. Since names are usually chosen to be mnemonic, however, the user may wish to take some precautions in choosing FORTRAN names.

Many IMSL names consist of a root name that may have a prefix to indicate the type of the routine. For example, the IMSL single precision routine for computing Bessel functions of the first kind with real order has the name `BSJS`, which is the root name, and the corresponding IMSL double precision routine has the name `DBSJS`. Associated with these two routines are `B2JS` and `DB2JS`. `BSJS` is listed in the Alphabetical Index of Routines, but `DBSJS, B2JS,` and `DB2JS` are not. The user of `BSJS` must consider both names `BSJS` and `B2JS` to be reserved; likewise, the user of `DBSJS` must consider both names `DBSJS` and `DB2JS` to be reserved. The root names of *all* routines and named common blocks that are used by MATH/LIBRARY Special Functions and that do not have a numeral in the second position of the root name are listed in the Alphabetical Index of Routines. Some of the routines in this Index are not intended to be called by the user and so are not documented. The careful user can avoid any conflicts with IMSL names if the following rules are observed:

- Do not choose a name that appears in the Alphabetical Summary of Routines in the *User's Manual*, nor one of these names preceded by a `D`, `S_`, `D_`, `C_`, or `Z_`.
- Do not choose a name of three or more characters with a numeral in the second or third position.

These simplified rules include many combinations that are, in fact, allowable. However, if the user selects names that conform to these rules, no conflict will be encountered.

Deprecated Features and Deleted Routines

Automatic Workspace Allocation

FORTRAN subroutines that work with arrays as input and output often require extra arrays for use as workspace while doing computations or moving around data. IMSL routines generally do not require the user explicitly to allocate such arrays for use as workspace. On most systems the workspace allocation is handled transparently. The only limitation is the actual amount of memory available on the system.

On some systems the workspace is allocated out of a stack that is passed as a FORTRAN array in a named common block `WORKSP`. A very similar use of a workspace stack is described by Fox et al. (1978, pages 116–121). (For compatiblity with older versions of the IMSL Libraries, space is allocated from the `COMMON` block, if possible.)

The arrays for workspace appear as arguments in lower-level routines. For example, the IMSL routine `LSARG` (in Chapter 1, “Linear Systems”), which solves systems of linear equations, needs arrays for workspace. `LSARG` allocates arrays from the common area, and passes them to the lower-level routine `L2ARG` which does the computations. In the “Comments” section of the documentation for `LSARG`, the amount of workspace is noted and the call to `L2ARG` is described. This scheme for using lower-level routines is followed throughout the IMSL Libraries. The names of these routines have a “2” in the second position (or in the third position in double precision routines having a “`D`” prefix). The user can provide workspace explicitly and call directly the “2-level” routine, which is documented along with the main routine. In a very few cases, the 2-level routine allows additional options that the main routine does not allow.

Prior to returning to the calling program, a routine that allocates workspace generally deallocates that space so that it becomes available for use in other routines.

Changing the Amount of Space Allocated

This section is relevant only to those systems on which the transparent workspace allocator is not available.

By default, the total amount of space allocated in the common area for storage of numeric data is 5000 numeric storage units. (A numeric storage unit is the amount of space required to store an integer or a real number. By comparison, a double precision unit is twice this amount. Therefore, the total amount of space allocated in the common area for storage of numeric data is 2500 double precision units.) This space is allocated as needed for `INTEGER`, `REAL`, or other numeric data. For larger problems in which the default amount of workspace is insufficient, the user can change the allocation by supplying the FORTRAN statements to define the array in the named common block and by informing the IMSL workspace allocation system of the new size of the common array. To request 7000 units, the statements are

```
COMMON /WORKSP/ RWKSP
REAL RWKSP(7000)
CALL IWKIN(7000)
```

If an IMSL routine attempts to allocate workspace in excess of the amount available in the common stack, the routine issues a fatal error message that indicates how much space is needed and prints statements like those above to guide the user in allocating the necessary amount. The program below uses IMSL routine BSJS (See Chapter 6, “Bessel Funtions ” of this manual.) to illustrate this feature.

This routine requires workspace that is just larger than twice the number of function values requested.

```
            INTEGER    N
            REAL       BS(10000), X, XNU
            EXTERNAL   BSJS
      !                                  Set Parameters
            XNU = .5
            X   = 1.
            N   = 6000
            CALL BSJS (XNU, X, N, BS)
            END
```

Output

```
*** TERMINAL ERROR from BSJS.  Insufficient workspace for
***          current allocation(s). Correct by calling
***          IWKIN from main program with the three
***          following statements:  (REGARDLESS OF
***          PRECISION)
***                COMMON /WORKSP/  RWKSP
***                REAL RWKSP(12018)
***                CALL IWKIN(12018)

*** TERMINAL ERROR from BSJS.  The workspace requirement is
***          based on N =6000.
STOP
```

In most cases, the amount of workspace is dependent on the parameters of the problem so the amount needed is known exactly. In a few cases, however, the amount of workspace is dependent on the data (for example, if it is necessary to count all of the unique values in a vector). Thus, the IMSL routine cannot tell in advance exactly how much workspace is needed. In such cases, the error message printed is an estimate of the amount of space required.

Character Workspace

Since character arrays cannot be equivalenced with numeric arrays, a separate named common block WKSPCH is provided for character workspace. In most respects, this stack is managed in the same way as the numeric stack. The default size of the character workspace is 2000 character units. (A character unit is the amount of space required to store one character.) The routine analogous to IWKIN used to change the default allocation is IWKCIN.

The routines in the following list are being deprecated in Version 2.0 of MATH/LIBRARY Special Functions. A deprecated routine is one that is no longer used by anything in the library but is being included in the product for those users who may be currently referencing it in their application. However, any future versions of MATH/LIBRARY Special Functions will not include these routines. If any of these routines are being called within an application, it is recommended that you change your code or retain the deprecated routine before replacing this library with the next version. Most of these routines were called by users only when they needed to set up their own workspace. Thus, the impact of these changes should be limited.

```
G2DF
G2IN
G3DF
```

The following specific FORTRAN intrinsic functions are no longer supplied by IMSL. They can all be found in their manufacturer's FORTRAN runtime libraries. If any change must be made to the user's application as a result of their removal from the IMSL Libraries, it is limited to the redeclaration of the function from "external" to "intrinsic." Argument lists and results should be identical.

ACOS	CEXP	DATAN2	DSQRT
AINT	CLOG	DCOS	DTAN
ALOG	COS	DCOSH	DTANH
ALOG10	COSH	DEXP	EXP
ASIN	CSIN	DINT	SIN
ATAN	CSQRT	DLOG	SINH
ATAN2	DACOS	DLOG10	SQRT
CABS	DASIN	DSIN	TAN
CCOS	DATAN	DSINH	TANH

GAMS Index

Description

This index lists routines in MATH/LIBRARY Special Functions by a tree-structured classification scheme known as GAMS. Boisvert, Howe, Kahaner, and Springmann (1990) give the GAMS classification scheme. The classification scheme given here is Version 2.0. The first level of the classification scheme is denoted by a letter A thru Z as follows:

A.	Arithmetic, Error Analysis
B.	Number Theory
C.	Elementary and Special Functions
D.	Linear Algebra
E.	Interpolation
F.	Solution of Nonlinear Equations
G.	Optimization
H.	Differentiation and Integration
I.	Differential and Integral Equations
J.	Integral Transforms
K.	Approximation
L.	Statistics, Probability
M.	Simulation, Stochastic Modeling
N.	Data Handling
O.	Symbolic Computation
P.	Computational Geometry
Q.	Graphics
R.	Service Routines
S.	Software Development Tools
Z.	Other

There are seven levels in the classification scheme. Subclasses for levels 3, 5, and 7 are denoted by letters “a” thru “w”. Subclasses for levels 2, 4, and 6 are denoted by the numbers 1 thru 23.

The index given in the following pages lists routines in MATH/LIBRARY Special Functions within each GAMS subclass. The purpose of the routine appear alongside the routine name.

IMSL MATH/LIBRARY Special Functions

C .... ELEMENTARY AND SPECIAL FUNCTIONS (*search also class L5*)

C1 .... Integer-valued functions (e.g., floor, ceiling, factorial, binomial coefficient)

`BINOM`	Evaluates the binomial coefficient.
`FAC`	Evaluates the factorial of the argument.

C2 .... Powers, roots, reciprocals

`CBRT`	Evaluates the real or complex cube root.

C3 .... Polynomials

C3a .... Orthogonal

`INITS`	Initializes the orthogonal series so the function value is the number of terms needed to insure the error is no larger than the requested accuracy.

C3a2 ..... Chebyshev, Legendre

`CSEVL`	Evaluates the *N*-term Chebyshev series.

C4 .... Elementary transcendental functions

`CACOS`	Evaluates the complex arc cosine.
`CARG`	Evaluates the argument of a complex number.
`CASIN`	Evaluates the complex arc sine.
`CATAN`	Evaluates the complex arc tangent.
`CATAN2`	Evaluates the complex arc tangent of a ratio.
`COSDG`	Evaluates the cosine for the argument in degrees.
`COT`	Evaluates the real or complex cotangent.
`SINDG`	Evaluates the sine for the argument in degrees.

C4b .... Exponential, logarithmic

`ALNREL`	Evaluates the natural logarithm of one plus the argument.
`CLNREL`	Evaluates the principal value of the complex natural logarithm of one plus the argument.
`EXPRL`	Evaluates the real or complex exponential function factored from first order.
`LOG10`	Evaluates the principal value of the real or complex common logarithm.

C4c .... Hyperbolic, inverse hyperbolic

`ACOSH`	Evaluates the real or complex arc hyperbolic cosine.
`ASINH`	Evaluates the real or complex arc hyperbolic sine.
`ATANH`	Evaluates the arc hyperbolic tangent..
`CSINH`	Evaluates the complex hyperbolic sine.
`CTANH`	Evaluates the complex hyperbolic tangent.
`TAN`	Evaluates the real or complex tangent.

C5 .... Exponential and logarithmic integrals

`ALI`	Evaluates the logarithmic integral.

`CHI`	Evaluates the hyperbolic cosine integral.
`CI`	Evaluates the cosine integral.
`CIN`	Evaluates a function closely related to the cosine integral.
`CINH`	Evaluates a function closely related to the hyperbolic cosine integral.
`E1`	Evaluates the exponential integral for arguments greater than zero and the Cauchy principal value of the integral for arguments less than zero.
`EI`	Evaluates the exponential integral for arguments greater than zero and the Cauchy principal value for arguments less than zero.
`ENE`	Evaluates the exponential integral of integer order for arguments greater than zero scaled by EXP(X).
`SHI`	Evaluates the hyperbolic sine integral.
`SI`	Evaluates the sine integral.

C7......... Gamma

C7a....... Gamma, log gamma, reciprocal gamma

`ALGAMS`	Returns the logarithm of the absolute value of the gamma function and the sign of gamma.
`ALNGAM`	Evaluates the real or complex natural logarithm of the absolute value of the gamma function.
`GAMMA`	Evaluates the real or complex gamma function.
`GAMR`	Evaluates the reciprocal real or complex gamma function.
`POCH`	Evaluates a generalization of Pochhammer's symbol.
`POCH1`	Evaluates a generalization of Pochhammer's symbol starting from the first order.

C7b....... Beta, log beta

`ALBETA`	Evaluate the log of the real or complex beta function, $\ln \beta(a,b)$.
`BETA`	Evaluates the real or complex beta function.

C7c....... Psi function

`PSI`	Evaluates the logarithmic derivative of the gamma function for a real or complex argument.

C7e....... Incomplete gamma

`CHIDF`	Evaluates the chi-squared distribution function.
`CHIIN`	Evaluates the inverse of the chi-squared distribution function.
`GAMDF`	Evaluates the gamma distribution function.
`GAMI`	Evaluates the incomplete gamma function.
`GAMIC`	Evaluates the complementary incomplete gamma function.
`GAMIT`	Evaluates the Tricomi form of the incomplete gamma function.

C7f....... Incomplete beta

`BETAI`	Evaluates the incomplete beta function ratio.
`BETDF`	Evaluates the beta probability distribution function.

`BETIN` Evaluates the inverse of the beta distribution function.

C8......... Error functions

C8a....... Error functions, their inverses, integrals, including the normal distribution function

`ANORDF` Evaluates the standard normal (Gaussian) distribution function.

`ANORIN` Evaluates the inverse of the standard normal (Gaussian) distribution function.

`CERFE` Evaluates the complex scaled complemented error function.

`ERF` Evaluates the error function.

`ERFC` Evaluates the complementary error function.

`ERFCE` Evaluates the exponentially scaled complementary error function.

`ERFCI` Evaluates the inverse complementary error function.

`ERFI` Evaluates the inverse error function.

C8b....... Fresnel integrals

`FRESC` Evaluates the cosine Fresnel integral.

`FRESS` Evaluates the sine Fresnel integral.

C8c....... Dawson's integral

`DAWS` Evaluates Dawson function.

C10....... Bessel functions

C10a..... J, Y, *H*(1); *H*(2)

C10a1... Real argument, integer order

`BSJ0` Evaluates the Bessel function of the first kind of order zero.

`BSJ1` Evaluates the Bessel function of the first kind of order one.

`BSJNS` Evaluates a sequence of Bessel functions of the first kind with integer order and real or complex arguments.

`BSY0` Evaluates the Bessel function of the second kind of order zero.

`BSY1` Evaluates the Bessel function of the second kind of order one.

C10a2... Complex argument, integer order.

`BSJNS` Evaluates a sequence of Bessel functions of the first kind with integer order and real or complex arguments.

C10a3... Real argument, real order

`BSJS` Evaluates a sequence of Bessel functions of the first kind with real order and real positive arguments.

`BSYS` Evaluates a sequence of Bessel functions of the second kind with real nonnegative order and real positive arguments.

C10a4 ... Complex argument, real order

`CBJS`	Evaluates a sequence of Bessel functions of the first kind with real order and complex arguments.
`CBYS`	Evaluates a sequence of Bessel functions of the second kind with real order and complex arguments.

C10b..... I, K

C10b1 ... Real argument, integer order

`BSI0`	Evaluates the modified Bessel function of the first kind of order zero.
`BSI0E`	Evaluates the exponentially scaled modified Bessel function of the first kind of order zero.
`BSI1`	Evaluates the modified Bessel function of the first kind of order one.
`BSI1E`	Evaluates the exponentially scaled modified Bessel function of the first kind of order one.
`BSINS`	Evaluates a sequence of modified Bessel functions of the first kind with integer order and real or complex arguments.
`BSK0`	Evaluates the modified Bessel function of the third kind of order zero.
`BSK0E`	Evaluates the exponentially scaled modified Bessel function of the third kind of order zero.
`BSK1`	Evaluates the modified Bessel function of the third kind of order one.
`BSK1E`	Evaluates the exponentially scaled modified Bessel function of the third kind of order one.

C10b2 ... Complex argument, integer order

`BSINS`	Evaluates a sequence of modified Bessel functions of the first kind with integer order and real or complex arguments.

C10b3 ... Real argument, real order

`BSIES`	Evaluates a sequence of exponentially scaled Modified Bessel functions of the first kind with nonnegative real order and real positive arguments.
`BSIS`	Evaluates a sequence of Modified Bessel functions of the first kind with real order and real positive arguments.
`BSKES`	Evaluates a sequence of exponentially scaled modified Bessel functions of the third kind of fractional order.
`BSKS`	Evaluates a sequence of modified Bessel functions of the third kind of fractional order.

C10b4 ... Complex argument, real order

`CBIS`	Evaluates a sequence of Modified Bessel functions of the first kind with real order and complex arguments.
`CBKS`	Evaluates a sequence of Modified Bessel functions of the second kind with real order and complex arguments.

C10c .....Kelvin functions

`AKEI0`	Evaluates the Kelvin function of the second kind, kei, of order zero.
`AKEI1`	Evaluates the Kelvin function of the second kind, kei, of order one.
`AKEIP0`	Evaluates the Kelvin function of the second kind, kei, of order zero.
`AKER0`	Evaluates the Kelvin function of the second kind, ker, of order zero.
`AKER1`	Evaluates the Kelvin function of the second kind, ker, of order one.
`AKERP0`	Evaluates the derivative of the Kelvin function of the second kind, ker, of order zero.
`BEI0`	Evaluates the Kelvin function of the first kind, bei, of order zero.
`BEI1`	Evaluates the Kelvin function of the first kind, bei, of order one.
`BEIP0`	Evaluates the derivative of the Kelvin function of the first kind, bei, of order zero.
`BER0`	Evaluates the Kelvin function of the first kind, ber, of order zero.
`BER1`	Evaluates the Kelvin function of the first kind, ber, of order one.
`BERP0`	Evaluates the derivative of the Kelvin function of the first kind, ber, of order zero.

C10d.....Airy and Scorer functions

`AI`	Evaluates the Airy function.
`AID`	Evaluates the derivative of the Airy function.
`AIDE`	Evaluates the exponentially scaled derivative of the Airy function.
`AIE`	Evaluates the exponentially scaled Airy function.
`BI`	Evaluates the Airy function of the second kind.
`BID`	Evaluates the derivative of the Airy function of the second kind.
`BIDE`	Evaluates the exponentially scaled derivative of the Airy function of the second kind.
`BIE`	Evaluates the exponentially scaled Airy function of the second kind.

C14.......Elliptic integrals

`EJCN`	Evaluates the real or cmplex Jacobi elliptic function cn(x, m).
`EJDN`	Evaluates the real or complex Jacobi elliptic function dn(z, m).
`EJSN`	Evaluates the real or comple Jacobi elliptic function sn(x, m).
`ELE`	Evaluates the complete elliptic integral of the second kind $E(x)$.

`ELK`	Evaluates the complete elliptic integral of the kind $K(x)$.
`ELRC`	Evaluates an elementary integral from which inverse circular functions, logarithms and inverse hyperbolic functions can be computed.
`ELRD`	Evaluates Carlson's incomplete elliptic integral of the second kind `RD`(`X`, `Y`, `Z`).
`ELRF`	Evaluates Carlson's incomplete elliptic integral of the first kind `RF`(`X`, `Y`, `Z`).
`ELRJ`	Evaluates Carlson's incomplete elliptic integral of the third kind `RJ`(`X`, `Y`, `Z`, `RHO`).

C15....... Weierstrass elliptic functions

`CWPL`	Evaluates the Weierstrass *P*-function in the lemniscat case for complex argument with unit period parallelogram.
`CWPLD`	Evaluates the first derivative of the Weierstrass *P*-function in the lemniscatic case for complex argum with unit period parallelogram.
`CWPQ`	Evaluates the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.
`CWPQD`	Evaluates the first derivative of the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.

C17....... Mathieu functions

`MATCE`	Evaluates a sequence of even, periodic, integer order, real Mathieu functions.
`MATEE`	Evaluates the eigenvalues for the periodic Mathieu functions.
`MATSE`	Evaluates a sequence of odd, periodic, integer order, real Mathieu functions.

C19....... Other special functions

`SPENC`	Evaluates a form of Spence's integral.

L........... STATISTICS, PROBABILITY

L5......... Function evaluation (*search also class C*)

L5a....... Univariate

L5a1..... Cumulative distribution functions, probability density functions

`GCDF`	Evaluates a general continuous cumulative distribution function given ordinates of the density.

L5a1b... Beta, binomial

`BETDF`	Evaluates the beta probability distribution function.
`BINDF`	Evaluates the binomial distribution function.
`BINPR`	Evaluates the binomial probability function.

L5a1c... Cauchy, chi-squared

`CHIDF`	Evaluates the chi-squared distribution function.
`CSNDF`	Evaluates the noncentral chi-squared distribution function.

L5a1f .... F distribution

`FDF`	Evaluates the *F* distribution function.

L5a1g ... Gamma, general, geometric

`GAMDF`	Evaluates the gamma distribution function.

L5a1h ... Halfnormal, hypergeometric

`HYPDF`	Evaluates the hypergeometric distribution function.
`HYPPR`	Evaluates the hypergeometric probability function.

L5a1k ... Kendall *F* statistic, Kolmogorov-Smirnov

`AKS1DF`	Evaluates the distribution function of the one-sided Kolmogorov-Smirnov goodness of fit *D*+ or *D*– test statistic based on continuous data for one sample.
`AKS2DF`	Evaluates the distribution function of the Kolmogorov-Smirnov goodness of fit *D* test statistic based on continuous data for two samples.

L5a1n ... Negative binomial, normal

`ANORDF`	Evaluates the standard normal (Gaussian) distribution function.

L5a1p ... Pareto, Poisson

`POIDF`	Evaluates the Poisson distribution function.
`POIPR`	Evaluates the Poisson probability function.

L5a1t .... t distribution

`TDF`	Evaluates the Student's *t* distribution function.
`TNDF`	Evaluates the noncentral Student's *t* distribution function.

L5a2 ..... Inverse cumulative distribution functions, sparsity functions

`GCIN`	Evaluates the inverse of a general continuous cumulative distribution function given ordinates of the density.

L5a2b ... Beta, binomial

`BETIN`	Evaluates the inverse of the beta distribution function.

L5a2c .... Cauchy, chi-squared

`CHIIN`	Evaluates the inverse of the chi-squared distribution function.

L5a2f .... *F* distribution

`FIN`	Evaluates the inverse of the *F* distribution function.

L5a2n ... Negative binomial, normal, normal scores

`ANORIN`	Evaluates the inverse of the standard normal (Gaussian) distribution function.

L5a2t .... *t* distribution

`TIN`	Evaluates the inverse of the Student's *t* distribution function.

L5b .... Multivariate

L5b1 ..... Cumulative distribution functions, probability density functions

L5b1n ... Normal

	`BNRDF`	Evaluates the bivariate normal distribution function.

N........... DATA HANDLING

N1......... Input, output

	`IFNAN`	Checks if a value is NaN (not a number).

N4......... Storage management (e.g., stacks, heaps, trees)

	`IWKCIN`	Initializes bookkeeping locations describing the character workspace stack.
	`IWKIN`	Initializes bookkeeping locations describing the workspace stack.

R........... SERVICE ROUTINES

R1......... Machine-dependent constants

	`AMACH`	Retrieves single-precision machine constants.
	`DMACH`	Retrieves double precision machine constants.
	`IFNAN`	Checks if a value is NaN (not a number).
	`IMACH`	Retrieves integer machine constants.
	`UMACH`	Sets or retrieves input or output device unit numbers.

R3......... Error handling

	`ERSET`	Sets error handler default print and stop actions.
	`IERCD`	Retrieves the integer code for an informational error.

Alphabetical Summary of Routines

IMSL MATH/LIBRARY Special Functions

ACOS	17	Evaluates the complex arc cosine.
ACOSH	25	Evaluates the real or complex arc hyperbolic cosine.
AI	149	Evaluates the Airy function.
AID	152	Evaluates the derivative of the Airy function.
AIDE	157	Evaluates the exponentially scaled derivative of the Airy function.
AIE	154	Evaluates the exponentially scaled Airy function.
AKEI0	138	Evaluates the Kelvin function of the second kind, kei, of order zero.
AKEI1	147	Evaluates the Kelvin function of the second kind, kei, of order one.
AKEIP0	142	Evaluates the Kelvin function of the second kind, kei, of order zero.
AKER0	137	Evaluates the Kelvin function of the second kind, ker, of order zero.
AKER1	146	Evaluates the Kelvin function of the second kind, ker, of order one.
AKERP0	141	Evaluates the derivative of the Kelvin function of the second kind, ker, of order zero.
AKS1DF	201	Evaluates the distribution function of the one-sided Kolmogorov-Smirnov goodness of fit *D*+ or *D*– test statistic based on continuous data for one sample.
AKS2DF	204	Evaluates the distribution function of the Kolmogorov-Smirnov goodness of fit *D* test statistic based on continuous data for two samples.
ALBETA	71	Evaluates the natural logarithm of the complete beta function for positive arguments.

ALGAMS	57	Returns the logarithm of the absolute value of the gamma function and the sign of gamma.
ALI	36	Evaluates the logarithmic integral.
ALNGAM	55	Evaluates the real or complex function, ln $\lvert\gamma(x)\rvert$.
ALNREL	6	Evaluates $\ln(x + 1)$ for real or complex x.
AMACH	264	Retrieves single-precision machine constants.
ANORDF	206	Evaluates the standard normal (Gaussian) distribution function.
ANORIN	208	Evaluates the inverse of the standard normal (Gaussian) distribution function.
ASIN	16	Evaluates the complex arc sine.
ASINH	24	Evaluates $\sinh^{-1} x$ for real or complex x.
ATAN	18	Evaluates the complex arc tangent.
ATAN2	19	Evaluates the complex arc tangent of a ratio.
ATANH	27	Evaluates $\tanh^{-1} x$ for real or complex x.
BEI0	136	Evaluates the Kelvin function of the first kind, bei, of order zero.
BEI1	145	Evaluates the Kelvin function of the first kind, bei, of order one.
BEIP0	140	Evaluates the derivative of the Kelvin function of the first kind, bei, of order zero.
BER0	135	Evaluates the Kelvin function of the first kind, ber, of order zero.
BER1	144	Evaluates the Kelvin function of the first kind, ber, of order one.
BERP0	139	Evaluates the derivative of the Kelvin function of the first kind, ber, of order zero.
BETA	69	Evaluates the real or complex beta function, $\beta(a,b)$.
BETAI	73	Evaluates the incomplete beta function ratio.
BETDF	209	Evaluates the beta probability distribution function.
BETIN	212	Evaluates the inverse of the beta distribution function.
BI	150	Evaluates the Airy function of the second kind.
BID	153	Evaluates the derivative of the Airy function of the second kind.
BIDE	158	Evaluates the exponentially scaled derivative of the Airy function of the second kind.

BIE	155	Evaluates the exponentially scaled Airy function of the second kind.
BINDF	190	Evaluates the binomial distribution function.
BINOM	50	Evaluates the binomial coefficient.
BINPR	191	Evaluates the binomial probability function.
BNRDF	213	Evaluates the bivariate normal distribution function.
BSI0	98	Evaluates the modified Bessel function of the first kind of order zero.
BSI0E	104	Evaluates the exponentially scaled modified Bessel function of the first kind of order zero.
BSI1	100	Evaluates the modified Bessel function of the first kind of order one.
BSI1E	106	Evaluates the exponentially scaled modified Bessel function of the first kind of order one.
BSIES	118	Evaluates a sequence of exponentially scaled modified Bessel functions of the first kind with nonnegative real order and real positive arguments.
BSINS	111	Evaluates a sequence of modified Bessel functions of the first kind with integer order and real or complex arguments.
BSIS	117	Evaluates a sequence of modified Bessel functions of the first kind with real order and real positive arguments.
BSJ0	92	Evaluates the Bessel function of the first kind of order zero.
BSJ1	94	Evaluates the Bessel function of the first kind of order one.
BSJNS	109	Evaluates a sequence of Bessel functions of the first kind with integer order and real arguments.
BSJS	113	Evaluates a sequence of Bessel functions of the first kind with real order and real positive arguments.
BSK0	101	Evaluates the modified Bessel function of the third kind of order zero.
BSK0E	107	Evaluates the exponentially scaled modified Bessel function of the third kind of order zero.
BSK1	103	Evaluates the modified Bessel function of the third kind of order one.
BSK1E	108	Evaluates the exponentially scaled modified Bessel function of the third kind of order one.

BSKES	121	Evaluates a sequence of exponentially scaled modified Bessel functions of the third kind of fractional order.
BSKS	120	Evaluates a sequence of modified Bessel functions of the third kind of fractional order.
BSY0	95	Evaluates the Bessel function of the second kind of order zero.
BSY1	97	Evaluates the Bessel function of the second kind of order one.
BSYS	115	Evaluates a sequence of Bessel functions of the second kind with real nonnegative order and real positive arguments.
CARG	1	Evaluates the argument of a complex number.
CBIS	127	Evaluates a sequence of modified Bessel functions of the first kind with real order and complex arguments.
CBJS	123	Evaluates a sequence of Bessel functions of the first kind with real order and complex arguments.
CBKS	129	Evaluates a sequence of modified Bessel functions of the third kind with real order and complex arguments.
CBRT	2	Evaluates the cube root of a real or complex number $\sqrt[3]{x}$.
CBYS	125	Evaluates a sequence of Bessel functions of the second kind with real order and complex arguments.
COSH	21	Evaluates the complex hyperbolic cosine.
CERFE	80	Evaluates the complex scaled complemented error function.
CHI	43	Evaluates the hyperbolic cosine integral.
CHIDF	215	Evaluates the chi-squared distribution function.
CHIIN	217	Evaluates the inverse of the chi-squared distribution function.
CI	39	Evaluates the cosine integral.
CIN	40	Evaluates a function closely related to the cosine integral.
CINH	44	Evaluates a function closely related to the hyperbolic cosine integral.
COSDG	14	Evaluates the cosine for the argument in degrees.
COT	11	Evaluates cot x for real x.
CPSI	66	Evaluates the logarithmic derivative of the gamma function for a complex argument.
CSEVL	257	Evaluates the N-term Chebyshev series.

CSINH	21	Evaluates the complex hyperbolic sine.
CSNDF	219	Evaluates the noncentral chi-squared distribution function.
CTANH	23	Evaluates the complex hyperbolic tangent.
CWPL	173	Evaluates the Weierstrass *P*-function in the lemniscat case for complex argument with unit period parallelogram.
CWPLD	175	Evaluates the first derivative of the Weierstrass *P*-function in the lemniscatic case for complex argum with unit period parallelogram.
CWPQ	176	Evaluates the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.
CWPQD	177	Evaluates the first derivative of the Weierstrass *P*-function in the equianharmonic case for complex argument with unit period parallelogram.
DAWS	85	Evaluates Dawson function.
DMACH	266	Retrieves double precision machine constants.
E1	33	Evaluates the exponential integral for arguments greater than zero and the Cauchy principal value of the integral for arguments less than zero.
EI	32	Evaluates the exponential integral for arguments greater than zero and the Cauchy principal value for arguments less than zero.
EJCN	180	Evaluates the Jacobi elliptic function cn(x, m).
EJDN	182	This function evaluates the Jacobi elliptic function dn(x, m).
EJSN	178	Evaluates the Jacobi elliptic function sn(x, m).
ELE	165	Evaluates the complete elliptic integral of the second kind $E(x)$.
ELK	163	Evaluates the complete elliptic integral of the kind $K(x)$.
ELRC	170	Evaluates an elementary integral from which inverse circular functions, logarithms and inverse hyperbolic functions can be computed.
ELRD	167	Evaluates Carlson's incomplete elliptic integral of the second kind `RD`(`X`, `Y`, `Z`).
ELRF	166	Evaluates Carlson's incomplete elliptic integral of the first kind `RF`(`X`, `Y`, `Z`).

ELRJ	169	Evaluates Carlson's incomplete elliptic integral of the third kind RJ(X, Y, Z, RHO).
ENE	35	Evaluates the exponential integral of integer order for arguments greater than zero scaled by EXP(X).
ERF	76	Evaluates the error function.
ERFC	77	Evaluates the complementary error function.
ERFCE	79	Evaluates the exponentially scaled complementary error function.
ERFCI	83	Evaluates the inverse complementary error function.
ERFI	82	Evaluates the inverse error function.
ERSET	261	Set error handler default print and stop actions.
EXPRL	4	Evaluates $(e^x - 1)/x$ for real or complex x.
FAC	48	Evaluates the factorial of the argument.
FDF	222	Evaluates the F distribution function.
FIN	223	Evaluates the inverse of the F distribution function.
FRESC	86	Evaluates the cosine Fresnel integral.
FRESS	88	Evaluates the sine Fresnel integral.
GAMDF	225	Evaluates the gamma distribution function.
GAMI	59	Evaluates the incomplete gamma function.
GAMIC	61	Evaluates the complementary incomplete gamma function.
GAMIT	63	Evaluates the Tricomi form of the incomplete gamma function.
GAMMA	51	Evaluates the real or complex gamma function, $\Gamma(x)$.
GAMR	54	Evaluates the reciprocal of the real or complex gamma function, $1/\Gamma(x)$.
GCDF	233	Evaluates a general continuous cumulative distribution function given ordinates of the density.
GCIN	236	Evaluates the inverse of a general continuous cumulative distribution function given ordinates of the density.
HYPDF	194	Evaluates the hypergeometric distribution function.
HYPPR	196	Evaluates the hypergeometric probability function.
IERCD	262	Retrieves the integer code for an informational error.
IFNAN	266	Checks if a value is NaN (not a number).

IMACH	263	Retrieves integer machine constants.
INITS	256	Initializes the orthogonal series so the function value is the number of terms needed to insure the error is no larger than the requested accuracy.
IWKCIN	271	Initializes bookkeeping locations describing the character workspace stack.
IWKIN	270	Initializes bookkeeping locations describing the workspace stack.
LOG10	5	Evaluate the complex base 10 logarithm, $\log_{10} z$.
MATCE	244	Evaluates a sequence of even, periodic, integer order, real Mathieu functions.
MATEE	241	Evaluates the eigenvalues for the periodic Mathieu functions.
MATSE	248	Evaluates a sequence of odd, periodic, integer order, real Mathieu functions.
N1RTY	262	Retrieves the error type set by the most recently called IMSL routine.
POCH	66	Evaluates a generalization of Pochhammer's symbol.
POCH1	67	Evaluates a generalization of Pochhammer's symbol starting from the first order.
POIDF	197	Evaluates the Poisson distribution function.
POIPR	199	Evaluates the Poisson probability function.
PSI	64	Evaluates the real or complex psi function, $\psi(x)$.
SHI	42	Evaluates the hyperbolic sine integral.
SI	38	Evaluates the sine integral.
SINDG	13	Evaluates the sine for the argument in degrees.
SPENC	255	Evaluates a form of Spence's integral.
TAN	10	Evaluates tan z for complex z.
TDF	227	Evaluates the Student's t distribution function.
TIN	229	Evaluates the inverse of the Student's t distribution function.
TNDF	231	Evaluates the noncentral Student's t distribution function.
UMACH	267	Set or Retrieves input or output device unit numbers.

Appendix C: References

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