Stability of accretion using the effective metric

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Outline of the talk

- Effective metric
- (Linear) stability through the effective metric
- Example: The model by Frolov
- Stability through the effective potential
- Discussion

Effective metric: scalar field - nonlinear theory

$$\mathscr{L} = \mathscr{L}(W) \qquad W = g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi$$

EOM:

$$\mathcal{L}_W \Box\phi + \mathcal{L}_{WW}(\partial^\mu\phi)\partial_\mu W + \cancel{W\frac{\partial\mathcal{L}_W}{\partial\phi}} - \cancel{\frac{\partial\mathcal{L}}{\partial\phi}} = 0$$

$$\phi = \phi_0 + \phi_1 \quad \Longrightarrow \quad \left(\sqrt{-g}\left(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}\right)\phi_{1,\mu}\right)_{,\nu} = 0.$$

Background metric

$$\sqrt{-g}\left(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}\right) = \sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu},$$

(all quantities evaluated at the background sol.)

$$\left(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\phi_{1,\mu}\right)_{,\nu} = 0.$$

E. Goulart and SEPB (2011)

$$\sqrt{-g}\,(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}) = \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu},$$

In the linear case, $\mathscr{L}(W)=W$ **the effective metric reduces to the backgd. metric.**

$$(\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\phi_{1,\mu})_{,\nu} = 0. \quad \Longleftrightarrow \quad S_2 = \int \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\phi_{1,\mu}\phi_{1,\nu}d^4x.$$

$$\tilde{T}^{\mu\nu} = \frac{\delta S_2}{\delta g^{\mu\nu}} \quad \Rightarrow \quad \tilde{T}^{\mu}_{\nu} = \tilde{g}^{\mu\lambda}\phi_{1,\lambda}\phi_{1,\nu} - \frac{1}{2}\delta^{\mu}_{\nu}\tilde{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}$$

$$\tilde{\nabla}_{\mu}\tilde{T}^{\mu\nu} = 0.$$

Stability using the effective metric

(Moncrief, 1980, for the case of a non-self gravitating potential perfect fluid accreting onto a Schwarzschild black hole)

If $\widetilde{X}^{\mu}$ is a Killing vector,

$$\widetilde{\nabla}_{\mu}\left(\widetilde{X}^{\nu}\widetilde{T}^{\mu}_{\nu}\right)=0,$$

$$\partial_{\nu}\left(\sqrt{-\tilde{g}}\;\tilde{X}^{\mu}\tilde{T}_{\mu}^{\nu}\right)=0$$

$$\widetilde{X}^{\nu}=\delta^{\nu}_{t}$$

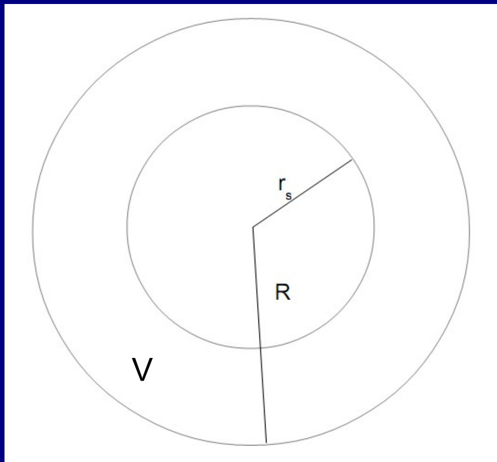
$$\widetilde{E}=\int_{V}\sqrt{-\widetilde{g}}\;\widetilde{T}^{t}_{t}d^{3}x.$$

$$\Longrightarrow$$

$$\frac{d\widetilde{E}}{dt}=-\int_{V}\left(\sqrt{-\widetilde{g}}\;\widetilde{T}^{r}_{t}\right)_{,r}d^{3}x.$$

$$\frac{d\widetilde{E}}{dt}=-\oint_{S}\sqrt{-\widetilde{g}}\left(\phi_{1,r}\phi_{1,t}\widetilde{g}^{rr}+(\phi_{1,t})^{2}\widetilde{g}^{rt}\right)dS_{r},$$

$$\widetilde{T}^{\mu}_{\nu}=\widetilde{g}^{\mu\lambda}\phi_{1,\lambda}\phi_{1,\nu}-\frac{1}{2}\delta^{\mu}_{\nu}\widetilde{g}^{\alpha\beta}\phi_{1,\alpha}\phi_{1,\beta}$$



$$\frac{d\widetilde{E}}{dt} = I_1 + I_2,$$

$$I_1 = -\oint_{S_R} \sqrt{-\tilde{g}}\left(\phi_{1,r}\phi_{1,t}\tilde{g}^{rr} + (\phi_{1,t})^2\tilde{g}^{rt}\right)\Big|_R \, dS_r,$$

$$I_2 = \oint_{S_{r_s}} \sqrt{-\tilde{g}}\left(\phi_{1,r}\phi_{1,t}\tilde{g}^{rr} + (\phi_{1,t})^2\tilde{g}^{rt}\right)\Big|_{r_s} \, dS_r,$$

r_s is the sonic horizon (see below)

Assume that

$$\sqrt{-\tilde{g}}\tilde{g}^{rr} \xrightarrow[r\to\infty]{} \sqrt{-g}g^{rr} = r^2 \sin\theta, \qquad \tilde{g}^{rt} \xrightarrow[r\to\infty]{} 0$$

Finite energy →

$$\phi_{1,t} \underset{r\to\infty}{\sim} \frac{a}{r^{\frac{3}{2}+\epsilon}}; \qquad \phi_{1,r} \sim \frac{a'}{r^{\frac{3}{2}+\epsilon}}, \quad \epsilon>0 \qquad \Rightarrow \qquad I_1 = 0 \quad (R\to\infty)$$

$$\widetilde{E} = \int_V \sqrt{-\tilde{g}}\, \widetilde{T}^t_t d^3x.$$

$$\frac{d\widetilde{E}}{dt} = I_2 = \oint_{S_{r_s}} \sqrt{-\tilde{g}} \left(\phi_{1,r}\phi_{1,t}\tilde{g}^{rr} + (\phi_{1,t})^2\tilde{g}^{rt}\right)\Big|_{r_s} dS_r,$$

$$\tilde{g}^{rr}(r_s) = 0 \quad \Longrightarrow \quad \frac{d\widetilde{E}}{dt} = \int \sqrt{-\tilde{g}}(\phi_{1,t})^2\tilde{g}^{rt}\Big|_{r_s} dS_r.$$

(assuming that there is a sonic horizon, see below)

This expression is valid for any sonic bh with the assumed symmetries. It gives the the time derivative of the energy of a (finite energy) perturbation in the 3-volume between r_s and infinity.

Stability $\Longleftrightarrow$ $\frac{d\widetilde{E}}{dt} \leq 0$

An example (Frolov, 2004)

$$\mathcal{L}(W) = \frac{1}{2}(W - A)^2.$$

It can be used to source the accelerated expansion as an effective cosmological constant (Arkani-Hamed et al, 2003)

$$ds^2 = f\,dt^2 - f^{-1}dr^2 - r^2 d\Omega^2,$$

$$f(r) = 1 - r_g/r$$

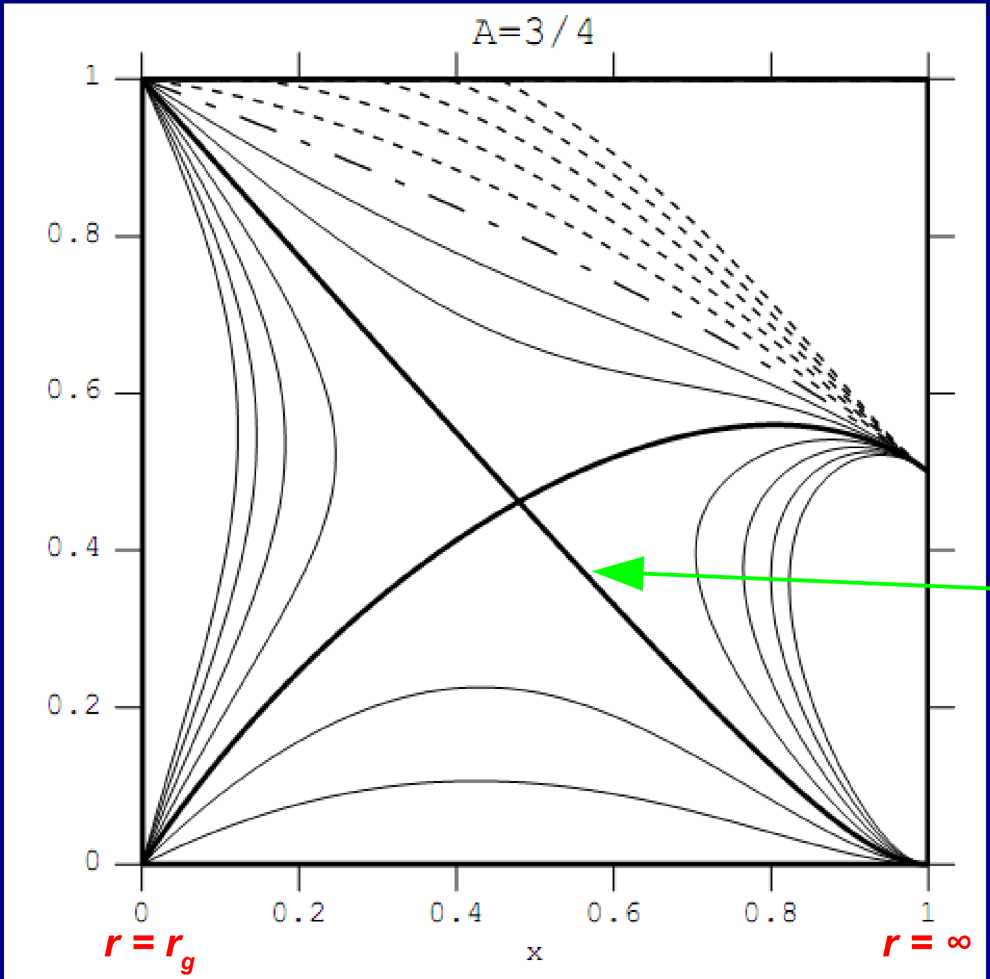
Steady state accretion
+ spherical symmetry

$$\phi_0 = t + \psi(r),$$

$$W = \frac{1 - (\partial_r^* \psi)^2}{f(r)},$$

$$\mathscr{L}_W \partial_r^* \psi = \alpha \frac{r_g^2}{r^2}$$

$$\partial_r^* \equiv f(r)\partial_r.$$



$$\mathscr{L}_W \partial_r^* \psi = \alpha \frac{r_g^2}{r^2}$$

$$\left(\frac{1-v^2}{x} - A\right)\frac{v}{(1-x)^2} = \alpha.$$

$v \equiv \partial_r^* \psi$ $\quad x \equiv f(r),$

There is only one solution that goes from infinity to r_g (independently of $0 \le A \le 1$) .

It is such that $\psi_{,r} > 0$.

The fluid interpretation

$X \leftrightarrow W$

$\mathscr{L}(X) = M^4 P(X)$ $\Rightarrow$ $T_{\mu\nu} = 2M^4 P'(X)\phi_{;\mu}\phi_{;\nu} + M^4 P(X) g_{\mu\nu}.$

which can be rewritten as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

$$\rho = M^4(2XP' - P), \quad p = M^4 P, \quad u_\mu = \frac{\phi_{;\mu}}{\sqrt{X}}.$$

$$w \equiv \frac{p}{\rho}, \quad c_s^2 \equiv \frac{dp}{d\rho} = \frac{p'}{\rho'}.$$

$$P(X) = \frac{1}{2}(X - A)^2,$$

$$w = \frac{X - A}{3X + A}, \quad c_s^2 = \frac{X - A}{3X - A}.$$

$$X = \frac{1 - (\partial_r^* \psi)^2}{f(r)},$$

The horizon is at $|\vec{u}| = c_s$ $\Rightarrow$ $r_s \simeq 3.8m$

for the chosen sol.

Back to the stability problem...

$$\frac{d\widetilde{E}}{dt} = \int \sqrt{-\tilde{g}}(\phi_{1,t})^2 \tilde{g}^{rt}\Big|_{r_s} dS_r.$$

$$\mathcal{L}(W) = \frac{1}{2}(W - A)^2.$$

$$\phi = t + \psi(r),$$

$$\sqrt{-g}\,(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}) = \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu},$$

$$\Downarrow$$

0

$$\frac{d\widetilde{E}}{dt} = -2\int r^2 sen\theta\,\psi_{,r}\phi_{1,t}^2 d\theta d\varphi\Big|_{r_s}$$

(only the sign of $\psi_{,r}$ is needed)

which is negative for the solution showed before

$\rightarrow$ **THE SYSTEM IS STABLE**

(C. A. Paz Rivasplata, J. M. Salim, SEPB, 2013)

Stability using the effective potential

$$(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\phi_{1,\mu})_{,\nu} = 0.$$

must be diagonalized

$$\sqrt{-g}\,(\mathcal{L}_W g^{\mu\nu} + 2\mathcal{L}_{WW}\phi_0^{;\mu}\phi_0^{;\nu}) = \sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu},$$

$$\begin{aligned} dt &= dT - \frac{\tilde{g}_{rt}}{\tilde{g}_{tt}} dR, \\ dr &= dR. \end{aligned}$$

$$\tilde{G}^{tt} = \frac{\tilde{g}^{tt}\tilde{g}^{rr} - \tilde{g}^{rt}}{\tilde{g}^{rr}},$$

$$\tilde{G}^{rr} = \tilde{g}^{rr},$$

$$\tilde{G}^{\theta\theta} = \tilde{g}^{\theta\theta},$$

$$\tilde{G}^{\varphi\varphi} = \tilde{g}^{\varphi\varphi}.$$

$$\tilde{G}^{rr}(r_s) = 0,$$

$$\partial_\mu(\sqrt{\tilde{G}}\tilde{G}^{\mu\nu}\partial_\nu\phi_1) = 0.$$

$$\phi_1 = \exp(-i\omega t)\beta(r)Y_{lm}(\theta,\varphi).$$

$$d\rho^* = F\mathcal{L}_W dr,$$

$$F = -\tilde{G}^{rr}$$

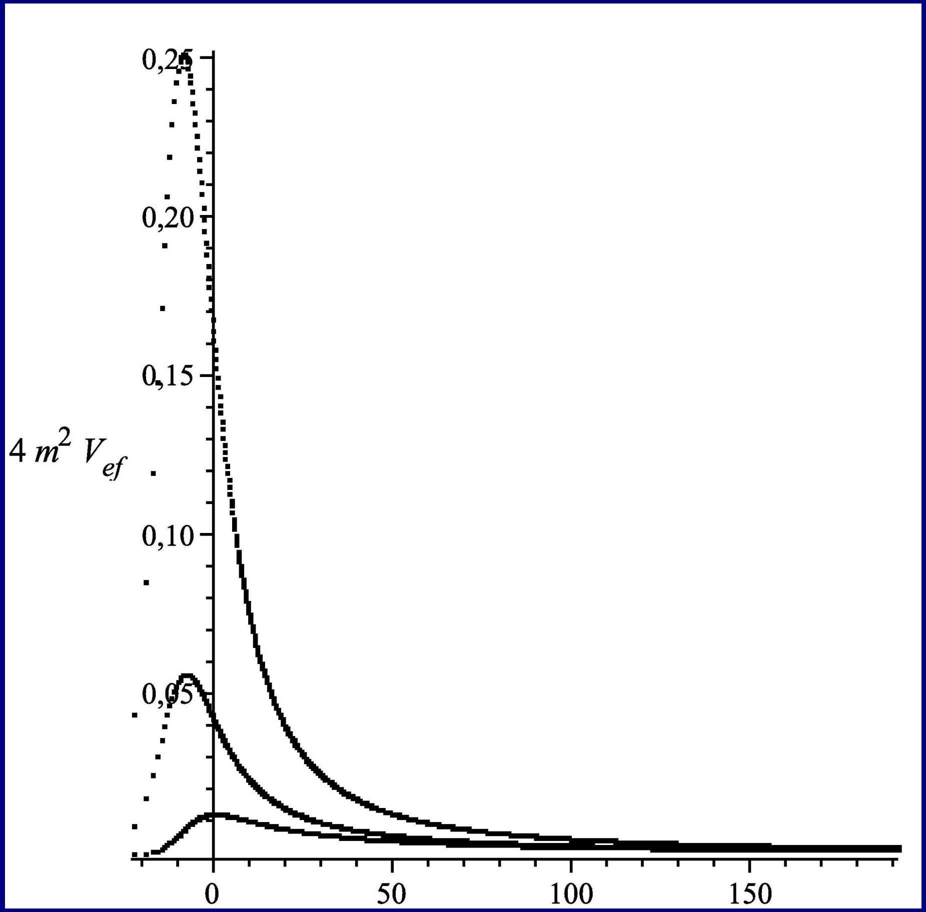
$\rho^* = \rho^*(r)$ must be calculated numerically

“tortoise” coordinates

After a long and straightforward calculation,

$$\frac{d^2\beta}{d\rho_*^2} + (\omega^2 - V_{ef}(r))\beta = 0$$

The explicit form of the function *ψ(r)* is needed.



(C. A. Paz Rivasplata, J. M. Salim, SEPB, 2013)

The positivity of the potential is a sufficient condition for stability (Detweiler and Ipser 1973).

Conclusions

- **The perturbations of a W-nonlinear field theory are governed (at the linear level) by an effective metric that depends on the nonlinearities of the theory and on the backgd. Solution.**
- **The time derivative of the energy of the perturbations can be evaluated as a surface integral, which depends on the effective metric.**
- **Using this integral, it was shown that the model by Frolov is stable.**
- **The integral method employed here requires much less work than the traditional effective potential method. In particular, the explicit form of the backgd. is not needed, only its derivative wrt *r*.**

- **It yields a definite result, while $V_{eff} > 0$ is a sufficient condition.**
- **It might be applied to systems with several dof, if in some regime the perturbations of one of them decouples from the rest of the perturbations.**

Example: Newtonian star

$$\partial_t \vec{v} = \vec{v} \times (\nabla \times \vec{v}) - \frac{1}{\rho} \nabla p - \nabla\left(\frac{1}{2} v^2 + \Phi\right).$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\rho = \rho_0 + \epsilon \rho_1$$

etc

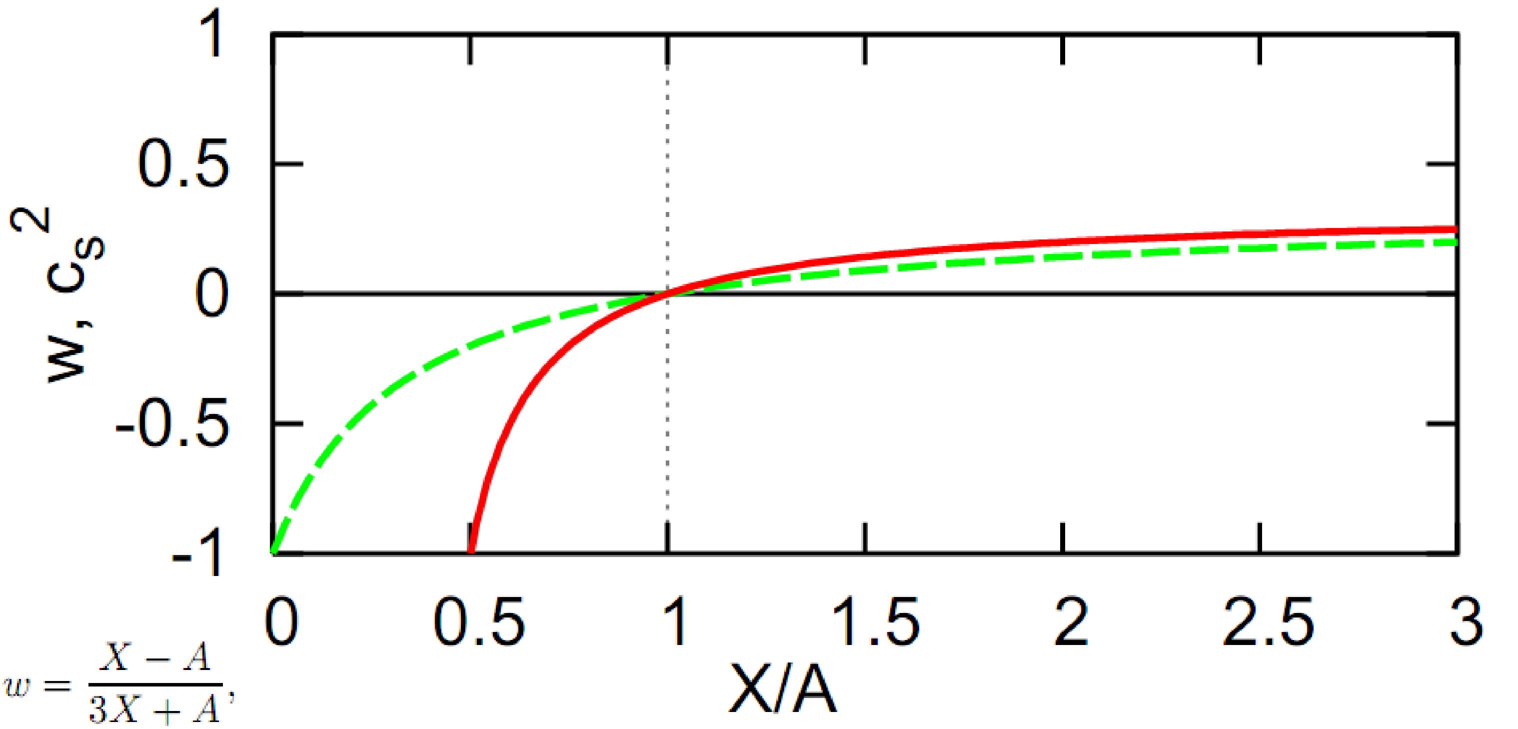
$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi_1) = -\frac{\rho_0}{\sqrt{-g}} \frac{D^{(0)}}{\partial t}\left(\frac{\Phi_1}{c^2}\right),$$

$$[\nabla^2 + k_J^2]\Phi_1 = k_J^2 \frac{D^{(0)}}{\partial t} \psi_1,$$

$$g^{\mu\nu} \equiv \frac{1}{\rho_0 c} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots & \cdot & \cdots \\ -v_0^i & \vdots & (c^2 \delta^{ij} - v_0^i v_0^j) \end{bmatrix},$$

$$k_J = \sqrt{\frac{4\pi G \rho_0}{c^2}}$$

$$\frac{D^{(0)}}{\partial t} = \partial_t + \vec{v}_0 \cdot \nabla.$$



$$w = \frac{X - A}{3X + A},$$

$$c_s^2 = \frac{X - A}{3X - A}.$$

$c^2 > 0$ guarantees stability only for ω >> Max (V_{eff})

$$\frac{d^2\beta}{dr^2} + \frac{1}{c_s^2}(\omega^2 - V_{ef}(r))\beta = 0,$$

$$\int_{-\infty}^{+\infty}\left[\left|\frac{\partial\phi_1}{\partial t}\right|^2+\left|\frac{\partial\phi_1}{\partial\rho^*}\right|^2+V_{ef}\left|\phi_1\right|^2\right]d\rho^*=\text{constant}$$

(No) Effective metric: scalar field - linear theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

$$\Box \phi + \frac{dV}{d\phi} = 0$$

$$\phi = \phi_0 + \varphi$$

in the EOM

$$\Box \phi_0 + \Box \varphi + \left.\frac{dV}{d\phi}\right|_0 + \left.\frac{d^2V}{d\phi^2}\right|_0 \varphi + O(\varphi^2) = 0$$

$$\Box \varphi + \left.\frac{d^2V}{d\phi^2}\right|_0 \varphi + O(\varphi^2) = 0$$

Eikonal: $\varphi(x) = \epsilon(x) e^{i\Phi(x)}$

in the EOM + $\lambda \longrightarrow 0$

$$\Box\varphi + \left.\frac{d^2V}{d\phi^2}\right|_0 \varphi + O(\varphi^2) = 0 \quad \Longrightarrow \quad \gamma^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) = 0$$

$$k_\mu \equiv (\partial_\mu\Phi) \quad \Longrightarrow \quad \gamma^{\mu\nu} k_\mu k_\nu = 0$$

$$\Downarrow$$

$$k^\lambda \nabla_\lambda k_\mu = 0$$

The high-energy excitations of φ follow geodesics of the background metric (in this case, Minkowski's) in the **linear theory**.

Effective metric: scalar field - nonlinear theory

$$\mathscr{L} = \mathscr{L}(X, \phi)$$

$$X \equiv \partial_\mu \phi \partial^\mu \phi$$

EOM:

$$\mathcal{L}_X \Box \phi + \mathcal{L}_{XX}(\partial^\mu \phi)\partial_\mu X + X \frac{\partial \mathcal{L}_X}{\partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\mathscr{L}_X \equiv \frac{\partial \mathscr{L}}{\partial X}$$

$$\phi = \phi_0 + \varphi$$

$$\varphi(x) = \epsilon(x) e^{i\Phi(x)}$$

$\lambda \longrightarrow 0$

$$\tilde{g}^{\mu\nu} k_\mu k_\nu = 0$$

$$k_\mu \equiv (\partial_\mu \Phi)$$

Effective metric

$$\tilde{g}^{\mu\nu} = [\mathcal{L}_X \gamma^{\mu\nu} + 2\mathcal{L}_{XX}(\partial^\mu \phi)(\partial^\nu \phi)]|_0$$

(C. Barcelo et al, 2011)

Background metric in Cartesian coords.

The high-energy excitations of φ follow geodesics of the **effective metric** in the **nonlinear theory**.

$$k^\lambda \tilde{\nabla}_\lambda k_\mu = 0$$