Points of General Relativistic Shock Wave Interaction are Regularity Singularities where Spacetime is Not Locally Flat

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Part I

Intuitive Introduction

About Shock Waves:

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- Shock waves form in fluids and gases, governed by **compressible Euler equations**.
- In General Relativity (GR), shock waves can be present in the matter content of spacetime.

Intuitive Introduction

FullImage_2005122151636_846.jpg (JPEG-Grafik, 500 × 326 Pixel) http://www.teamdroid.com/img-2/

Intuitive Introduction

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- Einstein tensor, $G^{\mu\nu}$, is determined by metric tensor.

$\longrightarrow$ How do shock waves effect the metric tensor?

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- At points of shock wave interaction: **No!**
  (R. and Temple, 2011)
  $\longrightarrow$ "Regularity Singularity"

Part II

Background: Shock Waves in General Relativity

Some mathematical terminology:

- regularity $\equiv$ handwavy expression refering to degree of differentiability
- $f(\cdot)$ is Lipschitz continuous $\equiv$ $f(\cdot)$ is point-wise differentiable with possibly discontinuous derivatives
- $C^{0,1}$ $\equiv$ set of Lipschitz continuous functions
- $C^{1,1}$ $\equiv$ set of functions whose derivatives are Lipschitz continuous

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$$\boxed{[T^{\mu\nu}]N_\nu = 0,}$$

  where
  - N^ν normal to Σ,
  - $[u] := u_L - u_R$ denotes the jump in u across Σ
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- Einstein equations, $G^{\mu\nu} = \kappa T^{\mu\nu}$, hold pointwise away from Σ.

Remark:

- A very important setting for shock waves in GR are the **Coupled Einstein Euler equations**,

$$\begin{aligned} G^{\mu\nu} &= \kappa T^{\mu\nu}, \\ \operatorname{div} T &= 0, \end{aligned}$$

where $\quad T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}, \quad$ (perfect fluid).

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- Shock waves can form in the relativistic Euler equations, $\operatorname{div} T = 0$, out of smooth initial data.

Part III

The Question of the Metric Regularity

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- Now, let a shock waves be present in $T^{\mu\nu}$
$\implies \quad T^{00}$ is discontinuous
$\implies \quad \frac{\partial B}{\partial r}$ is discontinuous

Conclusion:

- In some coordinates, the Einstein equations contain first order differential equations.
- Thus, if shock waves are present in $T^{\mu\nu}$, the metric is Lipschitz continuous ($C^{0,1}$), but not continuously differentiable (C^1). Thus, 2^{nd} order derivatives fail to exist point-wise.

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- Thus, if shock waves are present in $T^{\mu\nu}$, the metric is Lipschitz continuous ($C^{0,1}$), but not continuously differentiable (C^1). Thus, 2^{nd} order derivatives fail to exist point-wise.

Central Question:

Do there exist coordinates x^j such that the metric in the new coordinates, g_{ij}, is $C^{1,1}$ regular?

What's interesting about $C^{0,1}$ versus $C^{1,1}$ metric regularity?

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- $C^{1,1}$ is a common assumption in GR, e.g., $C^{1,1}$ regularity is required in Singularity Theorems (of Penrose, Hawking and Ellis).
- If one cannot smooth the metric to $C^{1,1}$, it cannot be locally Minkowski!
  ($\Rightarrow$ No observer in free-fall?!)

Part IV

The Metric Regularity Across a Single Shock Surface

“Israel’s Theorem“
(Israel 1966) (see also: Smoller and Temple 1994)

Suppose:

- $g_{\mu\nu}$ is $C^{0,1}$ across a single smooth surface Σ, but smooth away from Σ
- $g_{\mu\nu}$ solves the Einstein equations point-wise away from Σ

Then the following is equivalent:

- There exist a $C^{1,1}$ regular coordinate transformation, such that $g_{\alpha\beta}$ is $C^{1,1}$ regular, w.r.t. partial differentiation in the new coordinates x^{α}.
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Lesson:

Across a **single** shock one can always lift metric regularity to $C^{1,1}$!

Part V

The Metric Regularity at Points of Shock Wave Interaction

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NO, one cannot!
(R. & Temple, 2011)

Before we state our theorem, let me introduce the shock wave interaction we consider:

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Assumption on spacetime:

- Suppose spacetime M is **spherically symmetric**.

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- Suppose spacetime M is **spherically symmetric**.
- Assume Standard Schwarzschild Coordinates (=:SSC) exists around a point p, that is, assume the metric reads

$$g \;=\; -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega^2,$$

with

$$d\Omega^2 := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2.$$

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- Note: $\Sigma_i(t)$ is a 2-sphere with radius $x_i(t)$ and center $r = 0$.
- Instead of Σ_i it suffices to consider curves $\gamma_i(t) = (t, x_i(t))$, (so-called "shock curves").

Definition

$p \in M$ is a "point of regular shock wave interaction in SSC" if

- $\gamma_i(t) = (t, x_i(t))$, $i = 1, 2$, are smooth timelike curves defined on $t \in (-\epsilon, 0)$, which intersect in $p = \gamma_1(0) = \gamma_2(0)$.
- $T^{\mu\nu}$ is discontinuous across γ_1 and γ_2.
- Rankine Hugoniot conditions, $[T^{\mu\nu}]_i (N_i)_\nu = 0$, hold across each γ_i, (for $i = 1, 2$), and in the limit to $t \nearrow 0$.
- The SSC-metric, $g_{\mu\nu}$, is only $C^{0,1}$ across each shock curve and C^2 off and along them.
- Einstein equations hold point-wise off Σ_i.
- Shocks interact with distinct speeds, $\dot{x}_1(0) \neq \dot{x}_2(0)$.

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We expect this structure to be generic for radial shock waves in spherical symmetry, in agreement with the shock collisions simulated in (Vogler, 2011) and existence theory in (Groah & Temple, 2004).

Let's state our main theorem:

Theorem, (R. and Temple, 2011)

Assume p is "a point of regular shock wave interaction in SSC". Then there does **not** exist a $C^{1,1}$ coordinate transformation, defined in a neighborhood of p, such that both holds:

- The metric components are C^1 functions of the new coordinates.
- The metric has a nonzero determinant at p.

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Remark:

We only require two shock waves to be present before (or after) the interaction.

$\Longrightarrow$ We address many physical shock wave interaction, e.g.:

- two shock waves come in; two shock waves go out
- two shock waves come in; one shock and one rarefaction wave go out
- two compression waves come in; two shock waves go out

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Part VI

Proof of Theorem

Step 1:

- Assume there exist coordinates x^α, such that the transformed metric,

$$g_{\alpha\beta} = J^\mu_\alpha J^\nu_\beta g_{\mu\nu}, \tag{1}$$

is in C^1, where $J^\mu_\alpha = \frac{\partial x^\mu}{\partial x^\alpha}$ (Jacobian) and $g_{\mu\nu}$ metric in SSC.

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- Now, $g_{\alpha\beta}$ being in C^1 implies that, for all α, β, $\gamma \in \{0, ..., 3\}$,

$$[g_{\alpha\beta,\gamma}]_i = 0. \tag{2}$$

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- Now, substituting (1) into (2) leads to
$$\boxed{[J^\mu_{\alpha,\gamma}]_i J^\nu_\beta g_{\mu\nu} + [J^\nu_{\beta,\gamma}]_i J^\mu_\alpha g_{\mu\nu} + J^\mu_\alpha J^\nu_\beta [g_{\mu\nu,\gamma}]_i = 0.}$$

- Observe that above equation,

$$[J^{\mu}_{\alpha,\gamma}]_i J^{\nu}_{\beta} g_{\mu\nu} + [J^{\nu}_{\beta,\gamma}]_i J^{\mu}_{\alpha} g_{\mu\nu} + J^{\mu}_{\alpha} J^{\nu}_{\beta} [g_{\mu\nu,\gamma}]_i = 0, \qquad (3)$$

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  is necessary condition for smoothing the metric. It is linear in $[J^{\mu}_{\alpha,\gamma}]_i$.
- A lenghty computation shows that (3) together with the integrability condition,

$$[J^{\mu}_{\alpha,\beta}]_i = [J^{\mu}_{\beta,\alpha}]_i \ , \tag{4}$$

  has a unique solution $[J^{\mu}_{\alpha,\gamma}]_i$, given in terms of the jumps in the metric derivatives.

- In fact, the unique solution of (3) and (4) is given by:

$$
\begin{aligned}
[J^t_{0,t}]_i &= -\frac{1}{2}\left(\frac{[A_t]_i}{A}J^t_0 + \frac{[A_r]_i}{A}J^r_0\right); \quad [J^t_{0,r}]_i = -\frac{1}{2}\left(\frac{[A_r]_i}{A}J^t_0 + \frac{[B_t]_i}{A}J^r_0\right)\\
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\end{aligned}
\tag{5}
$$

- Notation:
  - $A_t := \frac{\partial A}{\partial t}$, ...
  - $\mu, \nu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$
  - J^t_0 denotes the $\mu = t$ and $\alpha = 0$ component of the Jacobian J^μ_α

Step (ii):

- The above solution $[J^\mu_{\alpha,\gamma}]_i$ is only defined on the shock curves. We now need to introduce functions, J^μ_α, whose derivatives have exactly the discontinuities prescribed by the solution of (3) and (4), that is $[J^\mu_{\alpha,\gamma}]_i$.

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- Such functions J^{μ}_{α} exist if and only if the Rankine-Hugoniot condition holds across both shocks. The functions J^{μ}_{α} are unique up to addition of C^1 functions.
- In fact, **every** function that meets (3) and (4) can be written in the **canonical form** stated in the following Lemma:

Lemma

If the RH jump condition hold, then there exists a set of functions $J^\mu_\alpha \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ that satisfies the smoothing condition (5) on $\gamma_i \cap \mathcal{N}$, $(i = 1, 2)$. All such J^μ_α assume the canonical form

$$\begin{aligned}
J^t_0(t,r) &= \sum_i \alpha_i(t)\,|x_i(t) - r| + \Phi(t,r)\,, \quad \alpha_i(t) = \frac{[A_r]_i\ \phi_i(t) + [B_t]_i\ \omega_i(t)}{4A \circ \gamma_i(t)}\,,\\
J^t_1(t,r) &= \sum_i \beta_i(t)\,|x_i(t) - r| + N(t,r)\,, \quad \beta_i(t) = \frac{[A_r]_i\ \nu_i(t) + [B_t]_i\ \zeta_i(t)}{4A \circ \gamma_i(t)}\,,\\
J^r_0(t,r) &= \sum_i \delta_i(t)\,|x_i(t) - r| + \Omega(t,r)\,, \quad \delta_i(t) = \frac{[B_t]_i\ \phi_i(t) + [B_r]_i\ \omega_i(t)}{4B \circ \gamma_i(t)}\,,\\
J^r_1(t,r) &= \sum_i \epsilon_i(t)\,|x_i(t) - r| + Z(t,r)\,, \quad \epsilon_i(t) = \frac{[B_t]_i\ \nu_i(t) + [B_r]_i\ \zeta_i(t)}{4B \circ \gamma_i(t)}\,, \qquad (6)
\end{aligned}$$

where

$$\phi_i = \Phi \circ \gamma_i, \quad \omega_i = \Omega \circ \gamma_i, \quad \zeta_i = Z \circ \gamma_i, \quad \nu_i = N \circ \gamma_i\,, \qquad (7)$$

and $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ have matching derivatives on each shock curve $\gamma_i(t)$,

$$[U_r]_i = 0 = [U_t]_i\,, \qquad (8)$$

for $U = \Phi, \Omega, Z, N$, $t \in (-\epsilon, 0)$.

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$$\left(\frac{\dot{x}_1\dot{x}_2}{A} + \frac{1}{B}\right)[B_r]_1[B_r]_2\,(\dot{x}_1 - \dot{x}_2)\left(\lim_{t\nearrow 0} Det\,\left(J^\mu_\alpha \circ \gamma_i(t)\right)\right) = 0.$$

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$$\left(\frac{\dot{x}_1\dot{x}_2}{A} + \frac{1}{B}\right)[B_r]_1[B_r]_2\,(\dot{x}_1 - \dot{x}_2)\left(\lim_{t\nearrow 0} Det\,\left(J^{\mu}_{\alpha}\circ\gamma_i(t)\right)\right) = 0.$$

- In above equation, all factors must be non-zero except the last one!

- We conclude that

$$\lim_{t \nearrow 0} Det\ (J^{\mu}_{\alpha} \circ \gamma_i(t)) = 0,$$

and thus, since $g_{\alpha\beta} = J^{\mu}_{\alpha} J^{\nu}_{\beta} g_{\mu\nu}$, we proved

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- We conclude that

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- This completes the proof. $\square$

Part VII

Summary and Consequences

- At points, p, of regular shock interaction in SSC the gravitational metric suffers a **non-removable** lack of C^1 regularity.

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- In particular, there exist (non-removable) distributional second order metric derivatives, but due to cancelation the spacetime curvature remains bounded.
- At p, spacetime is **not locally inertial**, that is, there do not exist coordinates x^j, such that the metric satisfies:
  - $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = diag(-1, 1, 1, 1)$,
  - $g_{ij,l}(p) = 0$,
  - $g_{ij,kl}$ are bounded on some neighborhood of p.

Part VIII

What's the physics behind all this?

- As spacetime fails to be locally inertial at points of shock wave collision and second order metric derivatives being distributional at such points, regularity singularities might be observable astrophysical objects.

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Thank you for your attention!

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