1 What is an elementary particle?

Two limitations on human view of nature:

- Finiteness of the maximum speed of propagation of interactions ($c \neq \infty$) $\Rightarrow$ Special Relativity

- Existence of a maximum acuracy in measurements of conjugated observables ($\hbar \neq 0$) $\Rightarrow$ Quantum Mechanics

Fundamental interactions are the domain where these two theories must match. But how?

Requirements:

1. Poincaré group MUST be unitarily represented in the Hilbert space that describes the situation;

2. Time evolution MUST preserve Poincaré symmetry.

Poincaré group:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} := P\left(\mathbf{\Lambda}, a\right)\left(x^{\mu}\right)$$

with

$$a^{\mu} \in \mathbb{R}$$
$$\mathbf{\Lambda}^{\mathsf{t}} \boldsymbol{\eta} \mathbf{\Lambda} = \boldsymbol{\eta}$$
$$\left(\eta_{\mu\nu}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\mathbf{\Lambda} \longrightarrow$ characterized by 6 independent parameters, $\omega^{\mu}{}_{\nu}$ $(\omega^{\mu}{}_{\nu} = -\omega_{\nu}{}^{\mu}) \Rightarrow$ Poincaré group transformations are fixed by 10 parameters ($\omega^{\mu}{}_{\nu}$ and a^{μ}).

Performing two successive transformations,

$$P(\mathbf{\Lambda}_2, a_2)\, P(\mathbf{\Lambda}_1, a_1) = P(\mathbf{\Lambda}_2\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 a_2 + a_1)$$

Unitary representation of Poincaré symmetry means that there must exist a set of unitary operators $U(\mathbf{\Lambda}, a)$ in one-to-one correspondence with each $P(\mathbf{\Lambda}, a)$, such that

$$U(\mathbf{\Lambda}_2, a_2)\, U(\mathbf{\Lambda}_1, a_1) = U(\mathbf{\Lambda}_2\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 a_2 + a_1)$$

Quantum Poincaré transformation:

$$U(\mathbf{\Lambda}, a) = \exp\left(\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i a_{\mu}P^{\mu}\right)$$

with ($J^{\mu\nu} = -J^{\nu\mu}$),

$$
\begin{aligned}
i\,[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}, \\
i\,[P^{\mu}, J^{\rho\sigma}] &= \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}, \\
[P^{\mu}, P^{\nu}] &= 0
\end{aligned}
$$

The Hamiltonian is identified with P^0 (generator of time translations). Labels of physical states come from the eigenvalues of operators that commute with $H = P^0$: these are the generators of spatial translations,

$$\mathbf{P} = \left\{P^1, P^2, P^3\right\}$$

and of rotations

$$\mathbf{J} = \left\{J^{23}, J^{31}, J^{12}\right\}.$$

Boost generators do not commute with H, but will have an important role in the following:

$$\mathbf{K} = \left\{J^{10}, J^{20}, J^{30}\right\}$$

Physical states

$$P^{\mu} |p, \sigma\rangle = p^{\mu} |p, \sigma\rangle$$

$\sigma \longrightarrow$ to be discussed in the following

Problem: what is the effect of $U(\mathbf{\Lambda}, a)$ on $|p, \sigma\rangle$ (or *what is* $U(\mathbf{\Lambda}, a)$)?

$$U(\mathbf{\Lambda}, a) |p, \sigma\rangle = U(\mathbf{\Lambda}, 0)\, U\left(\mathbf{1}, \mathbf{\Lambda}^{-1} a\right) |p, \sigma\rangle$$

$$= U(\mathbf{\Lambda}, 0) \exp\left(i \left(\mathbf{\Lambda}^{-1} a\right)_{\mu} P^{\mu}\right) |p, \sigma\rangle$$

$$= \exp\left(i \left(\mathbf{\Lambda}^{-1} a\right)_{\mu} p^{\mu}\right) U(\mathbf{\Lambda}, 0) |p, \sigma\rangle$$

So, we need to study the unitary representations of the *Lorentz group* ($a^{\mu} = 0$). Let's call $U(\mathbf{\Lambda}) := U(\mathbf{\Lambda}, 0)$. Using the algebra of the generators, it can be proven that

$$P^{\mu}\left(U\left(\mathbf{\Lambda}\right)|p,\sigma\rangle\right) = \Lambda^{\mu}{}_{\rho}p^{\rho}\left(U\left(\mathbf{\Lambda}\right)|p,\sigma\rangle\right)$$

$$\Rightarrow U\left(\mathbf{\Lambda}\right)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma'\sigma}\left(\mathbf{\Lambda},p\right)\left|\mathbf{\Lambda}p,\sigma'\right\rangle$$

New problem: to determinate $C_{\sigma'\sigma}\left(\mathbf{\Lambda},p\right)$. We will restrict ourselves to proper orthocronous Lorentz transformations (POLT) ($\det\mathbf{\Lambda} = 1$ and $\Lambda^{0}{}_{0} \geq 1$). The general case will be worked out later. These transformations preserve $p^{\mu}p_{\mu}$ and the sign of p^0 (for $p^2 \geq 0$).

All p^{μ} satisfying

$$p^2 = m^2,\; p^0 \geq 0$$

can be obtained from a given k^{μ} ($k^2 = m^2$, $k^0 \geq 0$) through a POLT

$$p^{\mu} = L^{\mu}{}_{\nu}\left(p\right)k^{\nu}$$

Definition (for $p \neq k$):

$$U\left(\boldsymbol{L}\left(p\right)\right)\left|k,\sigma\right\rangle = \sum_{\sigma'} C_{\sigma'\sigma}\left(\boldsymbol{L}\left(p\right),k\right)\left|p,\sigma'\right\rangle$$
$$\equiv \frac{1}{N_k\left(p\right)}\left|p,\sigma\right\rangle_k$$

The vectors $\left|p,\sigma\right\rangle_k$ constitute a basis as much as $\left|p,\sigma\right\rangle$. Now

$$\begin{aligned}U\left(\boldsymbol{\Lambda}\right)\left|p,\sigma\right\rangle_k &= N_k\left(p\right)U\left(\boldsymbol{\Lambda}\right)U\left(\boldsymbol{L}\left(p\right)\right)\left|k,\sigma\right\rangle = N\left(p\right)U\left(\boldsymbol{\Lambda}\boldsymbol{L}\left(p\right)\right)\left|k,\sigma\right\rangle\\ &= N_k\left(p\right)U\left(\boldsymbol{L}\left(\boldsymbol{\Lambda}p\right)\right)U\left(\underbrace{\boldsymbol{L}^{-1}\left(\boldsymbol{\Lambda}p\right)\boldsymbol{\Lambda}\boldsymbol{L}\left(p\right)}_{\boldsymbol{W}\left(\boldsymbol{\Lambda},p\right)}\right)\left|k,\sigma\right\rangle\\ &= N_k\left(p\right)U\left(\boldsymbol{L}\left(\boldsymbol{\Lambda}p\right)\right)U\left(\boldsymbol{W}\left(\boldsymbol{\Lambda},p\right)\right)\left|k,\sigma\right\rangle\end{aligned}$$

$\boldsymbol{W}$ satisfies

$$W^{\mu}{}_{\nu}k^{\nu} = k^{\mu}$$

$$U\left(\boldsymbol{W}\right)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}\left(\boldsymbol{W}\right)\left|k,\sigma'\right\rangle$$

The set of all Lorentz transformations $W^{\mu}{}_{\nu}$ which leave k^{μ} invariant is called *isotropy group* (or *little group*) of k^{μ}. If we know its representations $D_{\sigma'\sigma}$, the original problem is solved for the basis $|p,\sigma\rangle_k$ (up to normalization factors)

$$\begin{aligned} U\left(\boldsymbol{\Lambda}\right)|p,\sigma\rangle_k &= N_k\left(p\right)\sum_{\sigma'} D_{\sigma'\sigma}\left(\boldsymbol{W}\left(\boldsymbol{\Lambda},p\right)\right)U\left(\boldsymbol{L}\left(\boldsymbol{\Lambda}p\right)\right)\left|k,\sigma'\right\rangle \\ &= \frac{N_k\left(p\right)}{N_k\left(\boldsymbol{\Lambda}p\right)}\sum_{\sigma'} D_{\sigma'\sigma}\left(\boldsymbol{W}\left(\boldsymbol{\Lambda},p\right)\right)\left|\boldsymbol{\Lambda}p,\sigma'\right\rangle_k \end{aligned}$$

$$(N_k\left(p\right) = \sqrt{\frac{k^0}{p^0}})$$

- $m \neq 0$: choosing $k^{\mu} = (m,0,0,0) \rightarrow$ the isotropy group is the rotation group.

$D_{\sigma'\sigma}$ come from representations of the rotation group $\Rightarrow \sigma$ represents the property called *spin*.

- $m = 0$: choosing $k^\mu = (k, 0, 0, k) \longrightarrow$ the isotropy group is $ISO\,(2)$ (translations and rotations in 2 dimensions) $\Rightarrow \sigma$ represents the property called *helicity*.

Basis states in a Hilbert space where one has a unitary representation of the Poincaré group are associated (in one-to-one correspondence) to **elementary particles**. *The basis states are called* **one-particle states**.

What about the other components of the Lorentz group (det $\mathbf{\Lambda} = -1$, $\Lambda^0{}_0 \leq -1$)?

Fact: all Lorentz transformations can be decomposed (in the 4D representation) as a product of a POLT and $\mathcal{P}$, $\mathcal{T}$ or $\mathcal{PT}$, where

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \text{ space inversion}$$

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow \text{ time reversal}$$

In the Hilbert space, $\mathcal{P}$ has to correspond to a linear and unitary operator P and $\mathcal{T}$, to an antilinear and antiunitary operator T. Their effect in one particle states can be shown to be as follows:

- $m \neq 0$:

$$P\,|p,\sigma\rangle_k = \eta\,|\mathcal{P}p,\sigma\rangle_k \qquad \eta\text{: intrinsic parity (spin independent)}$$

$$T\,|p,\sigma\rangle_k = \zeta\,(-1)^{j-\sigma}\,|\mathcal{T}p,\sigma\rangle_k \qquad \zeta\text{: time reversal phase } (\zeta = 1)$$

- $m = 0$:

$$P\,|p,\sigma\rangle_k = \eta_\sigma \exp\left(\mp i\pi\sigma\right)|\mathcal{P}p,-\sigma\rangle_k \;\; \eta_\sigma\text{:} \; \begin{cases} \text{intrinsic parity;} \\ \text{helicity dependent;} \\ \mp \text{ if the sign of } p_2 \text{ is } \pm \end{cases}$$

$$T\,|p,\sigma\rangle_k = \zeta_\sigma \exp\left(\pm i\pi\sigma\right)|\mathcal{P}p,\sigma\rangle_k \;\; \zeta_\sigma\text{:} \; \begin{cases} \text{time reversal phase;} \\ \text{helicity dependent;} \\ \pm \text{ if the sign of } p_2 \text{ is } \pm \end{cases}$$

2 How do we take interaction into account?

Interaction $\rightarrow$ more than one particle!

Many particle states (drop the sufix "k"): tensor products of one-particle states

$$\left|p_1,\sigma_1;p_2,\sigma_2;...\right\rangle = \left|p_1,\sigma_1\right\rangle \otimes \left|p_2,\sigma_2\right\rangle \otimes ... \equiv \left|\alpha\right\rangle$$

The action of a POLT on such states is

$$\begin{aligned} U_0\left(\boldsymbol{\Lambda},a\right)\left|\alpha\right\rangle = U\left(\boldsymbol{\Lambda},a\right)\left|p_1,\sigma_1;p_2,\sigma_2;...\right\rangle &= \exp\left(ia_\mu\left(p_1^\mu+p_2^\mu+...\right)\right) \\ &\times \sqrt{\frac{\left(\boldsymbol{\Lambda}p_1\right)^0\left(\boldsymbol{\Lambda}p_2\right)^0...}{p_1^0p_2^0...}} \\ &\times \sum_{\sigma_1',\sigma_2'...} D^{(j_1)}_{\sigma_1'\sigma_1}\left(\boldsymbol{W}\left(\boldsymbol{\Lambda},p_1\right)\right) D^{(j_2)}_{\sigma_2'\sigma_2}\left(\boldsymbol{W}\left(\boldsymbol{\Lambda},p_2\right)\right)... \\ &\times \left|\boldsymbol{\Lambda}p_1,\sigma_1';\boldsymbol{\Lambda}p_2,\sigma_2';...\right\rangle \end{aligned}$$

$$|\alpha\rangle \longrightarrow \text{non-interacting state} \longrightarrow \text{eigenstate of an Hamiltonian } H_0$$

$$H_0 \, |\alpha\rangle = E_\alpha \, |\alpha\rangle \, , \qquad E_\alpha = p_1^0 + p_2^0 + \ldots$$

$$|\alpha\rangle \longrightarrow \text{non-normalizable state} \longrightarrow \text{physical states are superpositions}$$

$$|\psi\rangle = \int d\alpha g\,(\alpha)\,|\alpha\rangle$$

Interaction $\longrightarrow H = H_0 + V \Rightarrow |\alpha\rangle$ changes with time (possibility of creation and/or annihilation of particles!)

"In" $(|\alpha\rangle_+)$ and "out" $(|\alpha\rangle_-)$ states (Heisenberg picture) $\longrightarrow$ eigenstates of H which contain the particles described by α if the observations are done at $t = -\infty$ and $t = +\infty$, resp.

$$H \, |\alpha\rangle_\pm = E_\alpha \, |\alpha\rangle_\pm$$

What is the relationship between $|\alpha\rangle_+$ and $|\alpha\rangle_-$? Suppose that the clocks of two observers O and O' differ by a time interval of τ

$$t' = t - \tau$$

If O sees the system in the state $|\psi\rangle$, then O' will see it in the state (with $U \neq U_0$!)

$$\left|\psi'\right\rangle = U\left(\mathbf{1}, -\tau\right)|\psi\rangle = \exp\left(-iH\tau\right)|\psi\rangle$$

For physical states (superpositions of the basis states above) the definition of the "in" state requires that

$$\begin{aligned}\exp\left(-iH\tau\right)\int d\alpha g\left(a\right)|\alpha\rangle_- &= \int d\alpha e^{-iE_\alpha\tau} g\left(a\right)|\alpha\rangle_- \\ \xrightarrow[t\to-\infty]{} \int d\alpha e^{-iE_\alpha\tau} g\left(a\right)|\alpha\rangle &= \exp\left(-iH_0\tau\right)\int d\alpha g\left(a\right)|\alpha\rangle\end{aligned}$$

So, presuming the superposition, for $\tau_1 \to \infty$,

$$\exp\left(-iH_0\tau_1\right)|\alpha\rangle = \exp\left(-iH\tau_1\right)|\alpha\rangle_-$$

$$\Rightarrow |\alpha\rangle_{-} = \exp(iH\tau_1)\exp(-iH_0\tau_1)|\alpha\rangle \equiv \Omega(\tau_1)|\alpha\rangle$$

and, similarly, for $\tau_2 \to -\infty$,

$$\exp(-iH_0\tau_2)|\alpha\rangle = \exp(-iH\tau_2)|\alpha\rangle_{+}$$

$$\Rightarrow |\alpha\rangle_{+} = \exp(iH\tau_2)\exp(-iH_0\tau_2)|\alpha\rangle \equiv \Omega(\tau_2)|\alpha\rangle$$

Probability amplitude for the transition α $(t = -\infty) \to \beta$ $(t = +\infty)$: S *matrix*

$$S_{\beta\alpha} = {}_{-}\langle\beta|\alpha\rangle_{+}$$

$\left|S_{\beta\alpha}\right|^2$ is a physically measurable quantity!

It can be expressed (very formally!) also as

$$S_{\beta\alpha} = \langle\beta| \, S \, |\alpha\rangle$$

$$\begin{aligned} S &= \lim_{\substack{\tau_1 \to \infty \\ \tau_2 \to -\infty}} \Omega^\dagger (\tau_1) \, \Omega (\tau_2) \\ &= \lim_{\substack{\tau_1 \to \infty \\ \tau_2 \to -\infty}} \exp (iH_0\tau_1) \exp (-iH (\tau_1 - \tau_2)) \exp (-iH_0\tau_2) \end{aligned}$$

Poincaré covariance is expressed through

$$
\begin{aligned}
S_{\beta\alpha} &= {}_{-}\langle\beta|\alpha\rangle_{+} = {}_{-}\langle\beta| U^{\dagger}(\mathbf{\Lambda}, a) U(\mathbf{\Lambda}, a) |\alpha\rangle_{+} \\
&= \exp\left(ia_\mu \left(p_1^\mu + p_2^\mu + \ldots - p_1'^\mu - p_2'^\mu - \ldots\right)\right) \\
&\times \sqrt{\frac{(\mathbf{\Lambda}p_1)^0 (\mathbf{\Lambda}p_2)^0 \ldots \left(\mathbf{\Lambda}p_1'\right)^0 \left(\mathbf{\Lambda}p_2'\right)^0 \ldots}{p_1^0 p_2^0 \ldots p_1'^0 p_2'^0 \ldots}} \\
&\times \sum_{\bar{\sigma}_1, \bar{\sigma}_2 \ldots} D^{(j_1)}_{\bar{\sigma}_1 \sigma_1}\left(\boldsymbol{W}(\mathbf{\Lambda}, p_1)\right) D^{(j_2)}_{\bar{\sigma}_2 \sigma_2}\left(\boldsymbol{W}(\mathbf{\Lambda}, p_2)\right) \ldots \\
&\times \sum_{\bar{\sigma}_1', \bar{\sigma}_2' \ldots} D^{(j_1')*}_{\bar{\sigma}_1' \sigma_1'}\left(\boldsymbol{W}\left(\mathbf{\Lambda}, p_1'\right)\right) D^{(j_2')*}_{\bar{\sigma}_2' \sigma_2'}\left(\boldsymbol{W}\left(\mathbf{\Lambda}, p_2'\right)\right) \ldots \\
&\times S_{\mathbf{\Lambda}\beta\mathbf{\Lambda}\alpha} \\
&\Rightarrow S_{\beta\alpha} - \delta(\beta - \alpha) = -2\pi i M_{\beta\alpha} \delta^4\left(p_\beta - p_\alpha\right)
\end{aligned}
$$

The same formula could be obtained provided that the free states $|\alpha\rangle$ and $|\beta\rangle$

were transformed by $U_0(\mathbf{\Lambda}, a)$ and

$$U_0^{-1}(\mathbf{\Lambda}, a)\, S U_0(\mathbf{\Lambda}, a) = S \qquad \text{or} \qquad [S, U_0(\mathbf{\Lambda}, a)] = 0$$
$$\Rightarrow [H_0, S] = [\mathbf{P}_0, S] = [\mathbf{J}_0, S] = [\mathbf{K}_0, S] = 0$$

These conditions can be *proved* if we assume that $\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_I$ and that the matrix elements of $\mathbf{K}_I$ between eingenstates of H_0 are smooth functions of the energy.

How do we obtain $S_{\beta\alpha}$? From the defining equation for $|\alpha\rangle_\pm$,

$$\begin{aligned} H|\alpha\rangle_\pm = E_\alpha|\alpha\rangle_\pm &\Rightarrow (E_\alpha - H_0)|\alpha\rangle_\pm = V|\alpha\rangle_\pm \\ &\rightarrow |\alpha\rangle_\pm = |\alpha\rangle + (E_\alpha - H_0 \pm i\varepsilon)^{-1} V|\alpha\rangle_\pm \\ &= |\alpha\rangle + \int d\beta \frac{T^\pm_{\beta\alpha}}{E_\alpha - E_\beta \pm i\varepsilon}|\beta\rangle \end{aligned}$$

(*Lippmann-Schwinger equations*) with

$$T^{\pm}_{\beta\alpha} = \langle\beta| V |\alpha\rangle_{\pm}$$

From this, with some extra manipulations, it is possible to derive an integral formula for the S matrix:

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2\pi i \delta\left(E_\alpha - E_\beta\right) T^{\pm}_{\beta\alpha}$$

Born approximation (weak V):

$$S_{\beta\alpha} \simeq \delta(\beta - \alpha) - 2\pi i \delta\left(E_\alpha - E_\beta\right) \langle\beta| V |\alpha\rangle$$

This formula does not show Lorentz covariance explicitly. Let's use an equivalent (and more illuminating) approach: as we already saw

$$S = U(\infty, -\infty)$$

with

$$U(\tau, \tau_0) = \exp(iH_0\tau) \exp(-iH(\tau - \tau_0)) \exp(-iH_0\tau_0)$$

$\Rightarrow$

$$i\frac{d}{d\tau}U\left(\tau,\tau_0\right)=V\left(\tau\right)U\left(\tau,\tau_0\right)$$

$$V\left(t\right)=\mathsf{exp}\left(iH_0\tau\right)V\,\mathsf{exp}\left(-iH_0\tau\right)\qquad\text{(interaction picture)}$$

Integral equation:

$$U\left(\tau,\tau_0\right)=\mathbf{1}-i\int_{\tau_0}^{\tau}dtV\left(t\right)U\left(t,t_0\right)$$

Iterating and rearraging integration limits,

$$S=U\left(\infty,-\infty\right)=T\,\mathsf{exp}\left(-i\int_{-\infty}^{\infty}dtV\left(t\right)\right)$$

T means time-ordered product. Lorentz covariance can be explicitly achieved if the three conditions below are satisfied:

1. V is expressed as the integral of some local density,

$$V(t) = \int d^3x \mathcal{H}_I(\mathbf{x}, t)$$

2. The density is a Poincaré scalar

$$U_0(\mathbf{\Lambda}, a) \mathcal{H}_I(x) U_0^{-1}(\mathbf{\Lambda}, a) = \mathcal{H}_I(\mathbf{\Lambda}x + a)$$

3. (Causality condition) As time ordering is preserved under POLT for time and light-like separations $(x - x')^2 \geq 0$, we require that

$$\left[\mathcal{H}_I(x), \mathcal{H}_I\left(x'\right)\right] = 0, \qquad \text{for } \left(x - x'\right)^2 \leq 0$$

(the equality is due to possible problems when $x \to x'$).

3 Why do we need Quantum Field Theory?

How do we construct V? To begin to consider this question let's analyse the behavior of multiparticle states under exchange of two identical particles:

$$\Big|...p,\sigma;...p',\sigma';...\Big\rangle = \alpha\Big|...p',\sigma';...p,\sigma;...\Big\rangle$$
$$|\alpha|^2 = 1$$

Exchanging again

$$\alpha^2 = 1 \Rightarrow \alpha = \pm 1$$

(there are further considerations necessary to reach this conclusion, but it remains valid in 4D).

$$\alpha = +1 \rightarrow \text{ bosons}$$
$$\alpha = -1 \rightarrow \text{ fermions}$$

Multiparticle states have to be normalized consistently, according to the nature of the particles (boson or fermion): for a 2-particle state, for example (q denotes $\mathbf{p}$, σ, etc.):

$$\left\langle q_1' q_2' | q_1 q_2 \right\rangle = \delta\left(q_1' - q_1\right)\delta\left(q_2' - q_2\right) \pm \delta\left(q_1' - q_2\right)\delta\left(q_2' - q_1\right)$$

Let us define two very convenient operators, acting on the basis of free states:

Creation operators (bosons and fermions):

$$a_0^\dagger(q)\,|q_1, q_2, ...q_N\rangle \equiv |q, q_1, q_2, ...q_N\rangle$$

Annihilation operators (they are the adjoints of creation operators), defined for bosons $(+)$ and fermions $(-)$:

$$a_0(q)\,|q_1, q_2, ...\rangle = \sum_{r=1}^{N} (\pm 1)^{r+1}\,\delta(q - q_r)\,|q_1, ..., q_{r-1}, q_{r+1}, ...q_N\rangle$$

Multiparticle states can be constructed from the state with no particle (the *vacuum)*:

$$a_0^\dagger (q_1) \dots a_0^\dagger (q_N) |0\rangle = |q_1, q_2, \dots q_N\rangle$$

The vacuum is brought into the zero vector under the action of $a_0 (q)$:

$$a_0 (q) |0\rangle = 0$$

The definitions of a_0 and $a_0^\dagger$ imply

$$a_0 \left(q'\right) a_0^\dagger (q) \mp a_0^\dagger (q) a_0 \left(q'\right) = \delta \left(q - q'\right)$$
$$a_0 \left(q'\right) a_0 (q) \mp a_0 (q) a_0 \left(q'\right) = 0$$
$$a_0^\dagger \left(q'\right) a_0^\dagger (q) \mp a_0^\dagger (q) a_0^\dagger \left(q'\right) = 0$$

$\longrightarrow$ creation and annihilation operators that commute (anticommute) create and annihilate bosons (fermions).

Transformation of creation operators under Poincaré transformations:

$$U_0\left(\mathbf{\Lambda},\alpha\right) a_0^{\dagger}\left(\mathbf{p},\sigma\right) U_0^{-1}\left(\mathbf{\Lambda},\alpha\right) = e^{-i(\mathbf{\Lambda}p)\cdot\alpha}\sqrt{\frac{(\mathbf{\Lambda}p)^0}{p^0}} \times \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}\left(\boldsymbol{W}\left(\mathbf{\Lambda},p\right)\right) a_0^{\dagger}\left(\mathbf{p}_{\Lambda},\sigma\right)$$

(difficulty: under Poincaré transformations, the operator is multiplied by coefficients that depend on the momentum carried by the operator!)

Creation and annihilation operators could also be defined in terms of "in" and "out" states: this requires the introduction of "in" and "out" vacua

$$a_{\pm}\left(q\right)\left|0\right\rangle_{\pm} = 0$$
$$a_{\pm}^{\dagger}\left(q_1\right)\ldots a_{\pm}^{\dagger}\left(q_N\right)\left|0\right\rangle_{\pm} = \left|q_1, q_2, \ldots q_N\right\rangle_{\pm}$$

In this case, $a_{\pm}$ and $a_{\pm}^{\dagger}\left(q\right)$ must transform with the interacting representation of the Poincaré group, $U\left(\mathbf{\Lambda},\alpha\right)$.

Theorem 1 *Any operator O can be expressed as a sum of products of creation and annihilation operators:*

$$\begin{aligned} O = & \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int dq_1' ... dq_N' dq_1 ... dq_M \\ & \times a_0^{\dagger} \left(q_1' \right) ... a_0^{\dagger} \left(q_N' \right) a_0 \left(q_M \right) ... a_0 \left(q_1 \right) \\ & \times C_{NM} \left(q_1' ... q_N', q_1 ... q_M \right) \end{aligned}$$

Example: any additive operator

$$F \left| q_1, q_2, ... q_N \right\rangle = \left(f \left(q_1 \right) + ... \_ f \left(q_N \right) \right) \left| q_1, q_2, ... q_N \right\rangle$$

can be written as

$$F = \int dq a_0^{\dagger} \left(q \right) a_0 \left(q \right) f \left(q \right)$$

In particular,

$$H_0 = \int dq a_0^\dagger (q) \, a_0 (q) \, E (q)$$

Theorem 2 *If the Hamiltonian is expressed in terms of creation and annihilation operators like*

$$\begin{aligned} H = & \sum_{N=0}^{\infty} \sum_{M=0}^{\infty} \int dq_1' ... dq_N' dq_1 ... dq_M \\ & \times a_0^\dagger \left(q_1' \right) ... a_0^\dagger \left(q_N' \right) a_0 (q_M) ... a_0 (q_1) \\ & \times h_{NM} \left(q_1' ... q_N', q_1 ... q_M \right) \end{aligned}$$

with h_{NM} having the structure tiao

$$\begin{aligned} h_{NM} \left(q_1' ... q_N', q_1 ... q_M \right) = & \, \delta^3 \left(\mathbf{p}_1' + ... + \mathbf{p}_N' - \mathbf{p}_1 - ... - \mathbf{p}_M \right) \\ & \times \tilde{h}_{NM} \left(q_1' ... q_N', q_1 ... q_M \right) \end{aligned}$$

where $\tilde{h}_{NM}$ contains no delta function factors, then the S matrix satisfies the Cluster Decomposition Principle (CDP).

Wait! What says the CDP? Well, it says that *experiments that are sufficiently separated in space have unrelated results*. Quite reasonable!!!

The CDP implies a factorization structure for the S matrix, everytime its arguments become largely spatially separated:

$$S_{\beta_1+\beta_2+...+\beta_N,\alpha_1+\alpha_2+...+\alpha_N} = S_{\beta_1\alpha_1}S_{\beta_2\alpha_2}...S_{\beta_N\alpha_N}$$

where it is supposed that the particles indicated by α_i and β_i are close spatially to each other (in a *cluster*) and distant from other clusters.

H_0 is already in the proposed form

$$\begin{aligned} H_0 &= \int dq a_0^\dagger(q)\, a_0(q)\, E(q) \\ &= \int dq_1 dq_2 a_0^\dagger(q_1)\, a_0(q_2)\, \delta(q_1 - q_2)\, E(q_2) \end{aligned}$$

We need to consider V. It has to satisfy all the requirements stated previously and has to be built out of creation and annihilation operators. But how do we construct Poincaré scalars with them (remember their transformation laws, momentum dependent!)?

Proposal: we consider linear superpositions of creation and annihilation operators chosen in such a way as to transform as tensors under Poincaré transformations:

$$\begin{aligned} \varphi_{0,l}^{+}(x) &= \sum_\sigma \int d^3p u_l(x; \mathbf{p}, \sigma)\, a_0(\mathbf{p}, \sigma) \rightarrow \\ &\rightarrow U_0(\mathbf{\Lambda}, a)\, \varphi_{0,l}^{+}(x)\, U_0^{-1}(\mathbf{\Lambda}, a) = \sum_l L_{l\bar{l}}\left(\mathbf{\Lambda}^{-1}\right) \varphi_{0,\bar{l}}^{+}(\mathbf{\Lambda} x + a) \end{aligned}$$

$$\varphi^{-}_{0,l}(x) = \sum_{\sigma} \int d^3p v_l(x;\mathbf{p},\sigma)\, a^{\dagger}_0(\mathbf{p},\sigma) \rightarrow$$

$$\rightarrow U_0(\mathbf{\Lambda},a)\,\varphi^{-}_{0,l}(x)\,U_0^{-1}(\mathbf{\Lambda},a) = \sum_{l} L_{l\bar{l}}\left(\mathbf{\Lambda}^{-1}\right)\varphi^{-}_{0,\bar{l}}(\mathbf{\Lambda}x+a)$$

with $L_{l\bar{l}}(\mathbf{\Lambda})$ constituting a representation of the Lorentz group (by consistency). If we can do this, we can construct the interaction as

$$\mathcal{H}_I(x) = \sum_{NM} \sum\nolimits_{l'_1\ldots l'_N} \sum\nolimits_{l_1\ldots l_M} g_{l'_1\ldots l'_N,l_1\ldots l_M}$$
$$\times\, \varphi^{-}_{0,l'_1}(x)\ldots\varphi^{-}_{0,l'_N}(x)\,\varphi^{+}_{0,l_1}(x)\ldots\varphi^{+}_{0,l_M}(x)$$

with the g being Lorentz covariant

$$\sum_{l'_1\ldots l'_N} \sum_{l_1\ldots l_M} L_{l'_1\bar{l}'_1}\left(\mathbf{\Lambda}^{-1}\right)\ldots L_{l'_N\bar{l}'_N}\left(\mathbf{\Lambda}^{-1}\right) L_{l_1\bar{l}_1}\left(\mathbf{\Lambda}^{-1}\right)\ldots L_{l_M\bar{l}_M}\left(\mathbf{\Lambda}^{-1}\right)$$

$$\times g_{l'_1\ldots l'_N,l_1\ldots l_M} = g_{\bar{l}'_1\ldots\bar{l}'_N,\bar{l}_1\ldots\bar{l}_M}$$

What are the u_l's and v_l's? They **have** to satisfy

$$\sum_{\sigma} u_{\bar{l}}\left(\mathbf{\Lambda}x + b; \mathbf{p}_{\Lambda}, \bar{\sigma}\right) D^{(j_n)}_{\bar{\sigma}\sigma}\left(\boldsymbol{W}\left(\mathbf{\Lambda}, p\right)\right) = \sqrt{\frac{p^0}{\left(\mathbf{\Lambda}p\right)^0}}$$
$$\times \sum_{l} L_{\bar{l}l}\left(\mathbf{\Lambda}\right) \exp\left(i\left(\mathbf{\Lambda}p\right)\cdot b\right) u_{\bar{l}}\left(x; \mathbf{p}, \sigma\right)$$

and

$$\sum_{\sigma} v_{\bar{l}}\left(\mathbf{\Lambda}x + b; \mathbf{p}_{\Lambda}, \bar{\sigma}\right) D^{(j_n)*}_{\bar{\sigma}\sigma}\left(\boldsymbol{W}\left(\mathbf{\Lambda}, p\right)\right) = \sqrt{\frac{p^0}{\left(\mathbf{\Lambda}p\right)^0}}$$
$$\times \sum_{l} L_{\bar{l}l}\left(\mathbf{\Lambda}\right) \exp\left(-i\left(\mathbf{\Lambda}p\right)\cdot b\right) v_{\bar{l}}\left(x; \mathbf{p}, \sigma\right)$$

For $m \neq 0$, for example, translation + boost + rotation invariance $\Rightarrow$

$$u_l\left(x;\mathbf{p},\sigma\right) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{p^0}}e^{ip\cdot x}\sum_{\bar{l}} L_{l\bar{l}}\left(B\left(p\right)\right) u_{\bar{l}}\left(0,\sigma\right)$$

$$v_l\left(x;\mathbf{p},\sigma\right) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{m}{p^0}}e^{-ip\cdot x}\sum_{\bar{l}} L_{l\bar{l}}\left(B\left(p\right)\right) v_{\bar{l}}\left(0,\sigma\right)$$

$B\left(p\right) \rightarrow$ Boost that applied to a particle of mass m at rest takes it to a general four-momentum p^{μ}

$$\left.\begin{array}{c} u_l\left(0,\sigma\right) \\ v_l\left(0,\sigma\right) \end{array}\right\} \text{satisfy} \left\{\begin{array}{c} \sum_{\bar{\sigma}} u_{\bar{l}}\left(0,\bar{\sigma}\right)\mathbf{J}^{(j_n)}_{\bar{\sigma}\sigma} = \sum_{l} \vec{\mathcal{J}}_{\bar{l}l} u_l\left(0,\bar{\sigma}\right) \\ \sum_{\bar{\sigma}} v_{\bar{l}}\left(0,\bar{\sigma}\right)\mathbf{J}^{(j_n)*}_{\bar{\sigma}\sigma} = -\sum_{l} \vec{\mathcal{J}}_{\bar{l}l} v_l\left(0,\bar{\sigma}\right) \end{array}\right.$$

$\mathbf{J}^{(j_n)}_{\bar{\sigma}\sigma}$: rotation generators for spin n in the representation $D^{(j_n)}_{\bar{\sigma}\sigma}$ associated to the little group;

$\vec{\mathcal{J}}_{\bar{l}l}$: rotation generators in the representation $L_{l\bar{l}}(\mathbf{\Lambda})$ of the Lorenz group.

$\longrightarrow$ these equations fix $u_l\,(0,\sigma)$ up to a multiplicative constant (usual angular momentum algebra)!

Is it enough? Can we proceed and try to build V from $\varphi^{\pm}_{0,l}\,(x)$?

If we substitute the expressions that we found for $\varphi^{\pm}_{0,l}\,(x)$ in the expression for V we see that the result satisfies the CDP requirement. What about the causality condition? Notice that

$$\left[\varphi^{+}_{0,l}\,(x)\,,\phi^{+}_{0,l}\,(y)\right]_{\pm} = \left[\varphi^{-}_{0,l}\,(x)\,,\varphi^{-}_{0,l}\,(y)\right]_{\pm} = 0$$

but

$$\left[\varphi^{+}_{0,l}(x), \varphi^{-}_{0,l}(y)\right]_{\pm} = \frac{1}{(2\pi)^3} \sum_{\sigma} \int d^3 p u_l(\mathbf{p}, \sigma) v_l(\mathbf{p}, \sigma) e^{ip\cdot(x-y)} \neq 0$$

Solution: combine $\varphi^{\pm}_{0,l}(x)$ into

$$\phi_{0,l}(x) = \kappa_l \phi^{+}_{0,l}(x) + \lambda_l \phi^{-}_{0,l}(x)$$

and choose κ_l and λ_l such that, for $x - y$ spacelike

$$\left[\varphi_{0,l}(x), \varphi_{0,l}(y)\right]_{\pm} = \left[\varphi_{0,l}(x), \varphi^{\dagger}_{0,l}(y)\right]_{\pm} = 0$$

Last (apparent) obstacle: what if the particles carry other conserved quantum numbers? Then $\mathcal{H}_I(x)$ must commute with the corresponding observable.

Extension of notation: $(\mathbf{p},\sigma) \to (\mathbf{p},\sigma,n)$. For example, for eletric charge, Q

$$[Q, a_0(\mathbf{p},\sigma,n)] = -q(n)\, a_0(\mathbf{p},\sigma,n)$$
$$\left[Q, a_0^\dagger(\mathbf{p},\sigma,n)\right] = +q(n)\, a_0^\dagger(\mathbf{p},\sigma,n)$$

We need that $[Q,\mathcal{H}_I(x)] = 0$. This can be obtained if we require

$$\left[Q, \varphi_{0,l}(x)\right] = -q_l \varphi_{0,l}(x) \;\; (\Rightarrow \left[Q, \varphi_{0,l}^\dagger(x)\right] = q_l \varphi_{0,l}^\dagger(x))$$

and construct $\mathcal{H}_I(x)$ as a sum of products of fields $\varphi_{0,l_1}(x)\,\varphi_{0,l_2}(x)\ldots$ and adjoints $\varphi_{0,m_1}^\dagger(x)\,\varphi_{0,m_2}^\dagger(x)\ldots$ such that

$$q_{l_1} + q_{l_2} + \ldots - q_{m_1} - q_{m_2} - \ldots = 0$$

This means that all the creation operators $a_0^\dagger(\mathbf{p},\sigma,n)$ that appear in $\varphi_{0,l}(x)$ must be associated to a n such that $q(n) = -q_l$ and the annihilation operators, associated to a $\bar{n}$ satisfying $q(\bar{n}) = q_l$:

$$\varphi_{0,l}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{p^0}} \left\{ e^{ip\cdot x} u_l(\mathbf{p}, \sigma, n)\, a_0(\mathbf{p}, \sigma, n) \right.$$
$$\left. + e^{-ip\cdot x} v_l(\mathbf{p}, \sigma, \bar{n})\, a_0^\dagger(\mathbf{p}, \sigma, \bar{n}) \right\}$$

So, each $\varphi_{0,l}(x)$ creates and annihilates two different species of particles. Those labeled by $\bar{n}$ receive the name of *antiparticles*. The operator $\varphi_{0,l}(x)$ is what we call a *field operator*.

By construction, antiparticles **must** carry opposite values of the quantum numbers carried by particles.

All fields satisfy

$$\left(\Box + m^2\right) \varphi_{0,l}(x) = 0$$

and other subsidiary first order equations eventually implied by the particular representation of the Lorentz group that they carry. They also obey canonical equal-time commutation relations, implied by the corresponding relations obeyed by creation and annihilation operators

$$\left[\varphi_0\left(\mathbf{x},t\right),\dot{\varphi}_0\left(\mathbf{x}',t\right)\right]=i\delta\left(\mathbf{x}-\mathbf{x}'\right)$$
$$\left[\varphi_0\left(\mathbf{x},t\right),\varphi_0\left(\mathbf{x}',t\right)\right]=\left[\dot{\varphi}_0\left(\mathbf{x},t\right),\dot{\varphi}_0\left(\mathbf{x}',t\right)\right]=0$$

4 The S matrix and Green's functions

Simplest example of field: the **real massive scalar field** $\longrightarrow$ associated to the trivial representation of the rotation group ($\sigma=0$, $\mathbf{J}^{(0)}_{\bar{\sigma}\sigma}=0$).

Poincaré transformation:

$$U_0(\mathbf{\Lambda}, a)\,\varphi_0(x)\,U_0^{-1}(\mathbf{\Lambda}, a) = \varphi_0(\mathbf{\Lambda}x + a)$$

$\Rightarrow L_{l\bar{l}}\left(\mathbf{\Lambda}^{-1}\right) = 1$ and $\vec{\mathcal{J}}_{\bar{l}l} = 0$. Choosing $u(0) = v(0) = (2m)^{-1/2}$, we obtain

$$\varphi_0(x) = \int \frac{d^3p}{\sqrt{2p^0}\,(2\pi)^{3/2}} \left\{e^{ip\cdot x} a_0(\mathbf{p}) + e^{-ip\cdot x} a_0^\dagger(\mathbf{p})\right\} = \varphi_0^\dagger(x)$$

(no possibility of charged particles!). Taking the time derivative,

$$\dot{\varphi}_0(x) = i\int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{p^0}{2}} \left\{e^{ip\cdot x} a_0(\mathbf{p}) - e^{-ip\cdot x} a_0^\dagger(\mathbf{p})\right\}$$

Inverting these equations,

$$
\begin{aligned}
a_0(\mathbf{p}) &= \frac{1}{(2\pi)^{3/2}} \int d^3x e^{ip\cdot x} \left(\sqrt{\frac{p^0}{2}} \varphi_0(x) + i \sqrt{\frac{1}{2p^0}} \dot{\varphi}_0(x) \right) \\
&= \frac{-i}{(2\pi)^{3/2} \sqrt{2p^0}} \int d^3x e^{ip\cdot x} \overleftrightarrow{\partial_0} \varphi_0(x) \\
a_0^\dagger(\mathbf{p}) &= \frac{1}{(2\pi)^{3/2}} \int d^3x e^{-ip\cdot x} \left(\sqrt{\frac{p^0}{2}} \varphi_0(x) - i \sqrt{\frac{1}{2p^0}} \dot{\varphi}_0(x) \right) \\
&= \frac{i}{(2\pi)^{3/2} \sqrt{2p^0}} \int d^3x e^{-ip\cdot x} \overleftrightarrow{\partial_0} \varphi_0(x)
\end{aligned}
$$

with

$$
f \overleftrightarrow{\partial_0} g := (\partial_0 f)\, g - f\, (\partial_0 g)
$$

The same construction can be done for “in” and “out” creation and annihilation operators: it would express them in terms of "in" and "out" fields

$$a_{\pm}(\mathbf{p}) = \lim_{t\to\mp\infty} \frac{-i}{(2\pi)^{3/2}\sqrt{2p^0}} \int d^3x e^{ip\cdot x} \overleftrightarrow{\partial_0} \varphi_{\pm}(x)$$

$$a^{\dagger}_{\pm}(\mathbf{p}) = \lim_{t\to\mp\infty} \frac{i}{(2\pi)^{3/2}\sqrt{2p^0}} \int d^3x e^{-ip\cdot x} \overleftrightarrow{\partial_0} \varphi_{\pm}(x)$$

($a_{\pm}(\mathbf{p})$ and $a^{\dagger}_{\pm}(\mathbf{p})$ are t independent!). The fields $\varphi_{\pm}(x)$ are associated to $t = \mp\infty$. We can define *interacting fields* through a unitary transformation

$$\varphi(x) = U(t)\,\varphi_{\pm}(x)\,U^{-1}(x)$$

with

$$U(t) = \exp(-iHt)$$

This new field operator obeys the *Heisenberg equations*

$$\dot{\varphi} = \frac{1}{i}\left[\varphi, H\right]$$

and the same equal-time commutation relations as the free (or “in” and “out”) operators

$$\left[\varphi\left(\mathbf{x}, t\right), \dot{\varphi}\left(\mathbf{x}', t\right)\right] = i\delta\left(\mathbf{x} - \mathbf{x}'\right),$$
$$\left[\varphi\left(\mathbf{x}, t\right), \varphi\left(\mathbf{x}', t\right)\right] = \left[\dot{\varphi}\left(\mathbf{x}, t\right), \dot{\varphi}\left(\mathbf{x}', t\right)\right] = 0.$$

The Heisenberg equations can be obtained through the *canonical formalism*. We propose an action

$$S = \int d^4x \mathcal{L}\left(\varphi, \dot{\varphi}\right)$$

built from a *Lagrangian density* $\mathcal{L}(\varphi, \dot{\varphi})$ (scalar under Poincaré transformations); Define canonical momentum

$$\pi(x) = \frac{\partial \mathcal{L}(\varphi, \dot{\varphi})}{\partial \dot{\varphi}(x)} = \pi(\varphi, \dot{\varphi}) \rightarrow \dot{\varphi} = \dot{\varphi}(\varphi, \pi)$$

perform a Legendre transformation

$$\begin{aligned} \mathcal{H}(\varphi, \pi) &= \pi \dot{\varphi} - \mathcal{L}(\varphi, \dot{\varphi}(\varphi, \pi)) \\ &\rightarrow H = \int d^3x \mathcal{H} \end{aligned}$$

impose equal time commutation relations

$$\begin{aligned} \left[\varphi(\mathbf{x}, t), \pi\left(\mathbf{x}', t\right)\right] &= i\delta\left(\mathbf{x} - \mathbf{x}'\right), \\ \left[\varphi(\mathbf{x}, t), \varphi\left(\mathbf{x}', t\right)\right] &= \left[\pi(\mathbf{x}, t), \pi\left(\mathbf{x}', t\right)\right] = 0. \end{aligned}$$

and obtain the equations of motion in Hamiltonian form

$$\dot{\varphi} = \frac{1}{i}[\varphi, H]$$
$$\dot{\pi} = \frac{1}{i}[\pi, H]$$

For example, in the case of the real scalar field, we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - V(\varphi)$$

This gives the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + (\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi)\right)$$

and Heisenberg equations

$$\left(\Box + m^2\right)\varphi + V'(\varphi) = 0,$$

where V' denotes $\partial V(\varphi)/\partial\varphi$.

It is possible to conjecture that, under the assumptions of

- adiabatically switching off of the interaction;
- non-interaction between particles that constitute the “in” and “out” states;
- all quantities should be expressed in terms of the free (or “in” and “out”) fields;

the relation between "in" and "out" fields and the interacting field is (inside an expectation value)

$$\varphi(x) \underset{t\to\mp\infty}{\longrightarrow} Z^{1/2}\varphi_{\pm}(x)$$

with Z being a constant between 0 and 1.

Let us consider an alternative form for the S matrix element:

$$\begin{aligned}
S_{\beta\alpha} &= {}_{-}\langle\beta|\alpha\rangle_{+} = {}_{-}\left\langle p_1, p_2, ..., p_{N_-}|q_1, q_2, ..., q_{N_+}\right\rangle_{+} \\
&= {}_{-}\left\langle p_1, p_2, ..., p_{N_-}|a_{+}^{\dagger}(q_1)\,|q_2, ..., q_{N_+}\right\rangle_{+} \\
&= \int_t d^3x e^{-iq_1\cdot x} i\overleftrightarrow{\partial_0}\left[{}_{-}\left\langle p_1, p_2, ..., p_{N_-}|\varphi_{+}(x)\,|q_2, ..., q_{N_+}\right\rangle_{+}\right]
\end{aligned}$$

Choosing a large negative t we can write

$$\varphi_{+}(x) \underset{t\to-\infty}{\longrightarrow} Z^{-1/2}\varphi(x)$$

and performing (several) further manipulations,

$$S_{\beta\alpha} = \sum_{k=1}^{n} \sqrt{2p^0}\,(2\pi)^{3/2}\,\delta^3\left(\mathbf{p}_k - \mathbf{q}_1\right)\left[_-\left\langle p_1, p_2, ..., \widehat{p}_k, ...p_{N_-}|q_2, ..., q_{N_+}\right\rangle_+\right]$$
$$+\,iZ^{-1/2}\int d^4x e^{-iq_1\cdot x}\left(\Box + m^2\right)\left[_-\left\langle p_1, p_2, ..., p_{N_-}|\varphi\left(x\right)|q_2, ..., q_{N_+}\right\rangle_+\right]$$

We can repeat the above procedure with all momenta. The final result is

$$\begin{aligned} S_{\beta\alpha} = &\ \text{(disconnected terms)} \\ &+ \left(iZ^{-1/2}\right)^{N_-+N_+}\int d^4y_1...d^4y_{N_-}d^4x_1...d^4x_{N_+} \\ &\times \exp\left(i\sum_{k=1}^{N_-} p_k\cdot y_k - \sum_{r=1}^{N_+} q_r\cdot x_r\right) \\ &\times \left(\Box_{y_1} + m^2\right)...\left(\Box_{x_{N_+}} + m^2\right)\left[_-\left\langle 0|T\left(\varphi\left(y_1\right)...\varphi\left(x_{N_+}\right)\right)|0\right\rangle_+\right] \end{aligned}$$

$G_n(x_1, ..., x_n) = -\langle 0|T(\varphi(x_1)...\varphi(x_n))|0\rangle_+$ is the n-point *Green's function*

Z is absorbed in *renormalization*

$\Box + m^2$ denotes the existence of *poles* in the Fourier transform of the Green's function when $p_i^2 = m^2$

The obtainance of the Green's functions will be the main subject from now on