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Landscape complexity and dynamics in two simple statistical physics models

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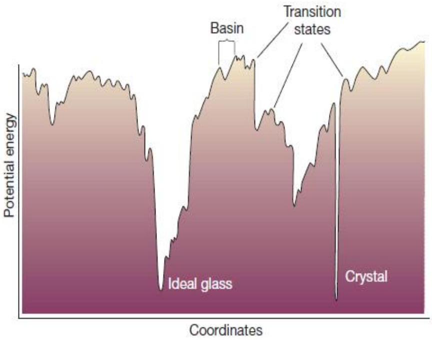
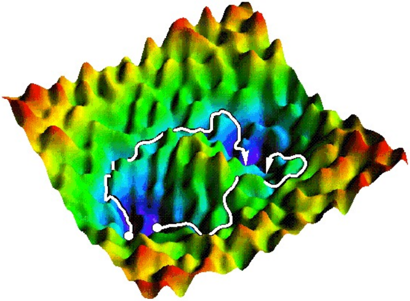
The Landscape approach to Complexity





Landscape versus Dynamical complexity

two simple models

Spherical spin glass
- "simple" *landscape*
- "complex" *dynamics*
- *quenched disorder*

XY mean field ferromagnet
- "complex" *landscape*
- "simple" *dynamics*
- *neither disorder nor frustration*

Seems paradoxical: what is the relation between landscape and dynamical complexity ?

The spherical spin glass (p=2)

Quenched disorder

$$E = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} S_i S_j \quad + \text{spherical constraint:} \quad \sum_{i=1}^{N} S_i^2 = N$$

$$-\sqrt{N} \leq s_i \leq \sqrt{N} \qquad i = 1, \ldots, N$$

J_{ij}'s: i.i.d. from GOE
$\langle J_{ij} \rangle = 0$ and $\langle J_{ij}^2 \rangle = J^2/N$

Equivalently

$$L = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} S_i S_j + \sigma \left(\sum_{i=1}^{N} S_i^2 - N \right)$$

The spherical spin glass

Landscape properties $\rightarrow$ complexity

The stationary point equations define an eigenvalue problem:

$$\frac{\partial L}{\partial s_k} = -\sum_{j=1}^{N} J_{kj} s_j + 2\sigma s_k = 0, \qquad k = 1, \ldots, N$$

2N stationary points with $\sigma_k = \frac{\lambda_k}{2} = -E_k/N$
λ_k: eigenvalues of GOE

The (constrained) Hessian at the stationary point *k* is

$$H_{kl} = \frac{\partial^2 L}{\partial s_K \partial s_l} = -J_{kl} + \lambda_k \delta_{kl}$$

At the k-th stationary point the eigenvalues are
$-\lambda_1 + \lambda_k < -\lambda_2 + \lambda_k < \ldots < -\lambda_N + \lambda_k$
there are two minima, two saddles of index 1, two saddles of index 2, ... , two maxima.

The spherical spin glass

Langevin dynamics

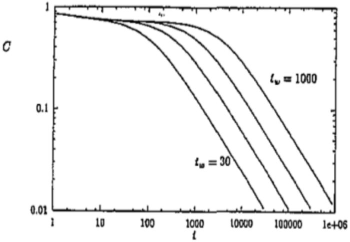
$$\frac{\partial s_k(t)}{\partial t} = -\frac{\partial E(t)}{\partial s_k(t)} - \lambda(t)s_k(t) + \xi_k(t)$$

$$= \sum_{j=1}^{N} J_{kj}s_j(t) - \lambda(t)s_k(t) + \xi_k(t)$$

$\xi_k(t)$: i.i.d. Gaussian variables with $\langle \xi_k(t) \rangle = 0$ and $\langle \xi_k(t)\xi_l(t') \rangle = 2T\delta_{kl}\delta(t-t')$.

For $N \to \infty$, the long time relaxation shows complex features such as *aging* in two-times quantities, e.g.

$$C(t_w, t_w+t) = \overline{\frac{1}{N}\sum_{i=1}^{N} S_i(t_w)S_i(t_w+t)} = \tilde{C}(t/t_w)$$

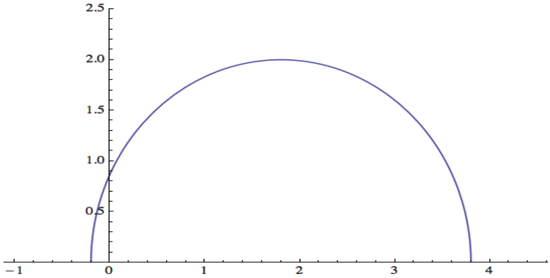


The spherical spin glass

Disorder average properties: the "shifted semicircle law"

$$H_{kl}(t) = \frac{\partial^2 E}{\partial s_K \partial s_l} + \lambda(t)\delta_{kl} = -J_{kl} + \lambda(t)\delta_{kl},$$

with $\lambda(t) = -2E(t)/N$. As the energy decreases, the distribution of eigenvalues shifts to the right at a progressively lower rate $\rightarrow$ slow relaxation.



The spherical spin glass

Langevin dynamics

For large but finite N the *average excess energy density*:
$[\Delta e(t)]_J = [e(t)]_J + \lambda_{max}/2$ decays in a nontrivial way:

$$[\Delta e(t)]_J \sim \begin{cases} \dfrac{3}{8t}, & t \ll N^{2/3} \\ N^{-2/3} F\left(\dfrac{t}{N^{2/3}}\right), & t \geq \mathcal{O}(N^{2/3}). \end{cases}$$

where

$$F(x) \sim \begin{cases} \dfrac{3}{8x}, & x \to 0 \\ \dfrac{A}{x^3}, & x \to \infty \end{cases}$$

Y. Fyodorov *et al.*, J. Stat. Mech. P11017 (2015)

The XY mean field ferromagnet

No disorder, no frustration

$$V = \frac{J}{2N} \sum_{i,j=1}^{N} \left[1 - \cos\left(\theta_i - \theta_j\right)\right] - h \sum_{i=1}^{N} \cos\theta_i$$

with $\theta_i \in [0, 2\pi]$. Defining:

$$m_x = \frac{1}{N} \sum_{i=1}^{N} \cos\theta_i$$

$$m_y = \frac{1}{N} \sum_{i=1}^{N} \sin\theta_i$$

then

$$V = \frac{JN}{2} \left(1 - m_x^2 - m_y^2\right) - hNm_x.$$

$$-h \leq v = V/N \leq \frac{J}{2} + \frac{h^2}{2}$$

The XY mean field ferromagnet

Landscape properties

Stationary point equations:

$$\frac{\partial V}{\partial \theta_i} = 0 \rightarrow (m_x + h)\sin\theta_i - m_y \cos\theta_i = 0, \qquad i = 1, \ldots, N$$

If $m_x + h$ and m_y are not simultaneously zero, then the critical points are all possible configurations with either $\theta_k = 0$ or π (Ising configurations).
If $0 \leq n_\pi \leq N$ is the number of angles equal to π in a c.p. configuration, then

$$m_x = 1 - \frac{2n_\pi}{N}$$

and the landscape has N energy levels:

$$v(n_\pi) = \frac{J}{2}\left[1 - \left(1 - \frac{2n_\pi}{N}\right)^2\right] - h\left(1 - \frac{2n_\pi}{N}\right)$$

The XY mean field ferromagnet

Landscape properties

The number of critical points with a fixed n_π is

$$C(n_\pi) = \begin{pmatrix} N \\ n_\pi \end{pmatrix} = \frac{N!}{n_\pi!(N - n_\pi)!}$$

There are $\sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} = 2^N$ critical points, an exponential number !

Stability of critical points:

$$\mathcal{H}_{ij} = \frac{\partial^2 V}{\partial\theta_i \partial\theta_j}$$

diagonal elements:

$$\mathcal{H}_{ii} = (m_x + h)\cos\theta_i + m_y \sin\theta_i - \frac{1}{N},$$

off-diagonal elements:

$$\mathcal{H}_{ij} = -\frac{1}{N}(\sin\theta_i \sin\theta_j + \cos\theta_i \cos\theta_j).$$

The XY mean field ferromagnet

Landscape properties

For $N \to \infty$ the hessian is diagonal, with

$$\lambda_i = (m_x + h)\cos\theta_i + m_y \sin\theta_i$$

At the critical points λ_i may assume only two possible values:

$$\begin{aligned} \lambda_i &= m_x + h & \quad i &= 1, \dots, N - n_\pi \\ \lambda_i &= -(m_x + h) & \quad i &= N - n_\pi + 1, \dots, N. \end{aligned}$$

The index of a critical point with n_π angles equal to π is:

$$\begin{aligned} index &= n_\pi & \quad if\ n_\pi &\leq \frac{N}{2} \\ index &= N - n_\pi & \quad if\ n_\pi &\geq \frac{N}{2} \end{aligned}$$

The XY mean field ferromagnet

Langevin Dynamics

$$V = \frac{J}{2N} \sum_{i,j=1}^{N} \left[1 - \cos\left(\theta_i(t) - \theta_j(t)\right)\right] - h \sum_{i=1}^{N} \cos\theta_i(t)$$

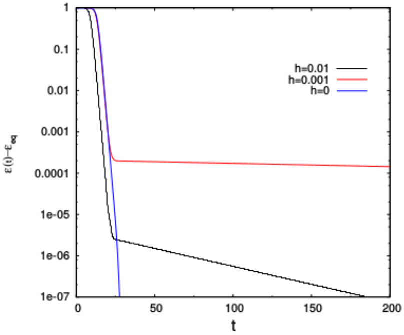
with $\theta_i(t) \in [0, 2\pi]$.

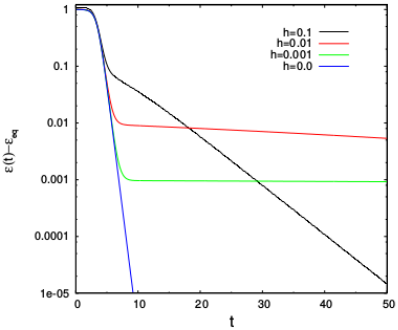
$$\begin{aligned} \frac{\partial \theta_k(t)}{\partial t} &= -\frac{\partial V}{\partial \theta_k(t)} + \eta_k(t) \\ &= \frac{J}{N} \sum_{i=1}^{N} \sin\left[\theta_i(t) - \theta_k(t)\right] - h \sin\theta_k(t) + \eta_k(t) \end{aligned}$$

$\eta_k(t)$ i.i.d. Gaussians with $\langle \eta_k t) \rangle = 0$ and $\langle \eta_k(t)\eta_l(t') \rangle = 2T\delta_{kl}\delta(t - t')$.

The XY mean field ferromagnet

Langevin Dynamics



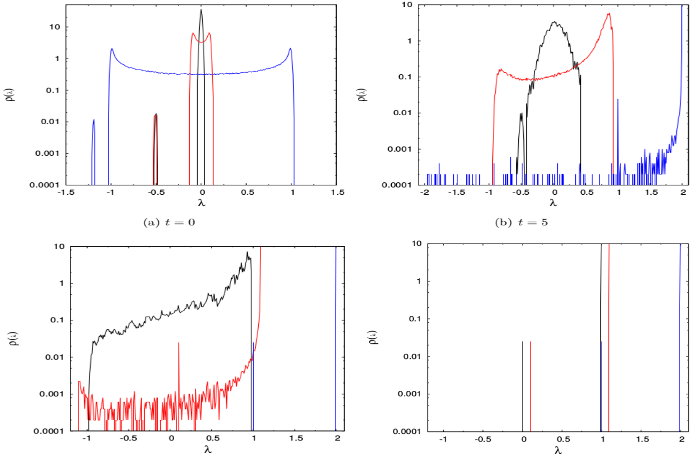


Energy relaxation for $T = 0$ and $N = 2 \times 10^6$.

Left: random initial conditions. Right: $\theta_k = 0$ if $k \leq N/2$, $\theta_k = 0.98\pi$ if $k > N/2$.

The XY mean field ferromagnet

Hessian eigenvalues distribution



(a) $t = 0$ (b) $t = 5$ (c) $t = 10$ (e) $t = 40$

Time evolution of hessian eigenvalue distribution for $T = 0.001$ and $N = 4000$ and 125 samples. $h = 0$ (black), $h = 0.1$ (red), $h = 1.0$ (blue). Random i.c.

Summary

- An exponential number of critical points with different stability characteristics (indexes) does not imply, a priori, a complex relaxation $\rightarrow$ XY ferromagnet with trivial hessian eigenvalue distribution.
- Quenched disorder is, clearly, a crucial ingredient as it produces non trivial statistical properties of the low energy states.
- But, also non disordered but frustrated systems show similar characteristics of slow complex dynamics.
- Is it possible to characterize the complexity of relaxation dynamics solely from the statistical properties of the hessian at the critical points of a landscape, whether quenched disorder is present or not ?

Thank you !