Superalgebras of (split-)division algebras and the split octonionic M-theory in $(6,5)$-signature

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Abstract

The connection of (split-)division algebras with Clifford algebras and supersymmetry is investigated. At first we introduce the class of superalgebras constructed from any given (split-)division algebra. We further specify which real Clifford algebras and real fundamental spinors can be reexpressed in terms of split-quaternions. Finally, we construct generalized supersymmetries admitting bosonic tensorial central charges in terms of (split-)division algebras. In particular we prove that split-octonions allow to introduce a split-octonionic M-algebra which extends to the $(6,5)$ signature the properties of the 11-dimensional octonionic M-algebras (which only exist in the $(10,1)$ Minkowskian and $(2,9)$ signatures).

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1 Introduction

2 Split-division algebras revisited

The construction of split-division algebras in terms of the Cayley-Dickson doubling procedure is reviewed in the Appendix. For later purposes it is useful to explicitly present here the (split-)division algebras structure constants, conjugations and norms in the case of quaternions ($\mathbb{H}$), split-quaternions ($\widetilde{\mathbb{H}}$), octonions ($\mathbb{O}$) and split-octonions ($\widetilde{\mathbb{O}}$). Complex ($\mathbb{C}$) and split-complex ($\widetilde{\mathbb{C}}$) numbers are immediately recovered as sub-algebra of, let's say, the split-quaternions.

Let us introduce at first the quaternions. The three imaginary quaternions $e_i \in \mathbb{H}$ ($i = 1, 2, 3$) satisfy the relations

$$e_i \cdot e_j \;=\; -\delta_{ij}\mathbf{1} + \epsilon_{ijk} e_k \tag{2.1}$$

(ϵ_{ijk} is totally antisymmetric tensor, normalized s.t. $\epsilon_{123} = 1$).

The conjugation and the norm are respectively given by

$$\begin{aligned} {e_i}^* &= -e_i, \\ N(e_i) &= 1. \end{aligned} \tag{2.2}$$

For what concerns the octonions, we can introduce them as as $\mathbb{O} = \mathbb{H}-$ (see Appendix). Therefore, the seven imaginary octonions E_i are recovered through the positions

$$\begin{aligned} E_i &= (e_i, 0) \\ E_{3+i} &= (0, e_i) \\ E_7 &= -(0, 1). \end{aligned} \tag{2.3}$$

They satisfy the relations

$$E_i \cdot E_j \;=\; -\delta_{ij}\mathbf{1} + C_{ijk} E_k, \tag{2.4}$$

while their conjugation and normalization are given by

$$\begin{aligned} {E_i}^* &= -E_i, \\ N(E_i) &= 1. \end{aligned} \tag{2.5}$$

In the above (2.4) formula the C_{ijk}'s are the totally antisymmetric octonionic structure constants, non-vanishing only for the triples*

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1. \tag{2.6}$$

*The seven imaginary octonions can be associated to the points of the seven-dimensional projective Fano's plane, while the triples correspond to the seven lines of this plane, see [**?**] for details.

With a similar procedure the split octonions can be expressed through the identificaion $\widetilde{\mathbb{O}} = \mathbb{H}+$. The seven imaginary split-octonions $\widetilde{E}_i$ are given, as before, by

$$\begin{aligned} \widetilde{E}_i &= (e_i, 0) \\ \widetilde{E}_{3+i} &= (0, e_i) \\ \widetilde{E}_7 &= -(0, 1). \end{aligned} \tag{2.7}$$

They satisfy the relations

$$\widetilde{E}_i \cdot \widetilde{E}_j = -\eta_{ij}\mathbf{1} + C_{ijk}\eta_{kr}\widetilde{E}_r, \tag{2.8}$$

together with

$$\begin{aligned} \widetilde{E}_i^* &= -\widetilde{E}_i, \\ N(\widetilde{E}_i) &= \eta_{ii}. \end{aligned} \tag{2.9}$$

In the above formulas η_{ij} denotes the diagonal matrix $(+++----)$ with three positive and four negative eigenvalues (normalized to ± 1).

The quaternionic subalgebra $\mathbb{H}$ of the split octonions is obtained by restricting the imaginary split-octonions $\widetilde{E}_i$ to the values $i = 1, 2, 3$.

On the other hand, the split-quaternionic subalgebra $\widetilde{\mathbb{H}}$ is recovered by taking any subset of three elements lying in the six other lines of the Fano's projective plane (namely, the triples (147), (165), (246), (257), (354) and (367)).

The split-quaternions subalgebra can be explicitly presented as follows, in terms of the three generators $\widetilde{e}_i$ $(i = 1, 2, 3)$,

$$\widetilde{e}_i \cdot \widetilde{e}_j = -\eta_{ij}\mathbf{1} + \epsilon_{ijk}\eta_{kr}\widetilde{e}_r, \tag{2.10}$$

with conjugation and norm given by

$$\begin{aligned} \widetilde{e}_i^* &= -\widetilde{e}_i, \\ N(\widetilde{e}_i) &= \eta_{ii}. \end{aligned} \tag{2.11}$$

η_{ij} is now the diagonal matrix $(--+)$.

The split quaternions admit a faithful representation in terms of 2×2 real matrices given by

$$\begin{aligned} &\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2.12}$$

The conjugate element of a generic split-quaternion $X \in \widetilde{\mathbb{H}}$ is represented by

$$X^* = -\tau_A X^T \tau_A. \tag{2.13}$$

3 Graded algebras from (split-)division algebras

The multiplication "$\cdot$" of a composition algebra $\mathbb{A}$ induces on $\mathbb{A}$ the structure of a (graded) algebra $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ of (anti)commutators. Namely, for $a, b \in \mathbb{A}$, we can introduce the algebra of graded brackets defined through

$$[a, b\} \;=\; ab + (-1)^{\epsilon_a \epsilon_b} ba, \tag{3.14}$$

where $\epsilon_{a,b} \equiv 0, 1 \; mod \, 2$ corresponds to the $\mathbb{Z}_2$ grading of the generators a, b respectively. The (anti)commutator algebra is a (graded) Lie algebra if the multiplication is associative. If the multiplication is alternative (see the Appendix), the (anti)commutator algebra is a (graded) Malcev algebra (see [2] for its definition).

The $\mathbb{Z}_2$ grading implies for $\mathbb{A}$ the decomposition $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1$ such that, for non-vanishing $[a, b\}$,

$$deg([a, b\}) \;=\; deg(a) + deg(b) \equiv \epsilon_a + \epsilon_b \; (mod \; 2) \tag{3.15}$$

We can easily list the set of admissible $\mathbb{Z}_2$ gradings for each one of the four division algebras (the $\mathbb{R}$ case is trivial). As a corollary, this gives us the list of the admissible superalgebras based on each division algebra. From the previous section results we know that the split-division algebras structure constants are recovered, up to a normalization factor, from the structure constants of their corresponding division algebra. For this reason the list of the admissible $\mathbb{Z}_2$ gradings (and associated superalgebras) of division algebras can also be regarded as the list of admissible $\mathbb{Z}_2$ gradings (and associated superalgebras) of the split-division algebras . The identity is necessarily an even (bosonic) element of the (super)algebra and corresponds to a central term. The (split) imaginary numbers close graded subalgebras of dimension 1 (for $\mathbb{C}$ and $\widetilde{\mathbb{C}}$), 3 (for $\mathbb{H}$ and $\widetilde{\mathbb{H}}$) and 7 (for $\mathbb{O}$ and $\widetilde{\mathbb{O}}$).

It is worth noticing that we can regard the (anti)commutators algebras induced by the composition law as *abstract* (super)algebras. In particular this implies that the $\mathbb{Z}_2$ superalgebra grading does not necessarily coincide with a $\mathbb{Z}_2$ grading of the composition law (which requires satisfying $deg(ab) = deg(a) + deg(b) \; mod \; 2$). This point can be better illustrated with an explicit example. Let's take the three imaginary quaternions e_i's. If we assign odd-grading (fermionic character) to e_1 and e_2, then e_3, appearing on the r.h.s. of the multiplication $e_1 \cdot e_2$, is necessarily even-graded (bosonic). On the other hand, the anticommutator $\{e_1, e_2\}$ is vanishing. As far as the anticommutators *alone* are concerned, we can consistently assign odd-grading to e_3 as well. In the following we will denote as "compatible" the restricted class of (super)algebras whose $\mathbb{Z}_2$ grading is an acceptable $\mathbb{Z}_2$ grading for the composition law.

The admissible $\mathbb{Z}_2$ gradings are expressed by the following table (the last column refers to the *compatible* superalgebras). We have

	$bosons/fermions$	$(super)algebra$	$compatibility$
$\mathbb{C}$, $\widetilde{\mathbb{C}}$	$1B+0F$	yes	yes
	$0B+1F$	yes	yes
$\mathbb{H}$, $\widetilde{\mathbb{H}}$	$3B+0F$	yes	yes
	$2B+1F$	no	$-$
	$1B+2F$	yes	yes
	$0B+3F$	yes	no
$\mathbb{O}$, $\widetilde{\mathbb{O}}$	$7B+0F$	yes	yes
	$6B+1F$	no	$-$
	$5B+2F$	no	$-$
	$4B+3F$	no	$-$
	$3B+4F^{(a)}$	no	$-$
	$3B+4F^{(b)}$	yes	yes
	$2B+5F$	no	$-$
	$1B+6F$	yes	no
	$0B+7F$	yes	no

(3.16)

There are two distinguished $3B+4F$ cases. The second one (b) corresponds to the three bosonic elements lying on one of the seven lines corresponding to the triples in (2.6). Without loss of generality, the three octonionic elements in the line can always be chosen as E_1, E_2 and E_3 ($\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$ for split-octonions). Without loss of generality the case (a) can be obtained by taking the three bosonic elements as E_1, E_2, E_4 ($\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_4$ for split-octonions, respectively). There is no superalgebra associated to the case (a), while a compatible superalgebra is found in the (b) case.

4 (Split-)division algebras, Clifford algebras and spinors

It is well-known that the Clifford algebras are related to the $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ associative division algebras. The $Cl(s,t)$ Clifford algebra is defined as the enveloping algebra generated by the gamma-matrices satisfying the relation

$$\Gamma_i\Gamma_j+\Gamma_j\Gamma_i \;=\; 2\eta_{ij}, \tag{4.17}$$

with η_{ij} a diagonal matrix of (s,t) signature (i.e. s positive, $+1$, and t negative, -1, diagonal entries).

The most general irreducible *real* matrix representation of a Clifford algebra is classified according to the property of the most general S matrix commuting with all the Γ's ($[S,\Gamma_i]=0$ for all i). If the most general S is a multiple of the identity, we get the normal ($\mathbb{R}$) case. Otherwise, S can be the sum of two matrices, the second one multiple of the square root of -1 (this is the almost complex, $\mathbb{C}$ case) or the linear combination of 4 matrices closing the quaternionic algebra (this is the $\mathbb{H}$ case). We obtain, for $s, t \leq 8$, the following table, see [?]

$s\backslash t$	0	1	2	3	4	5	6	7	8
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	${}^2\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$
1	${}^2\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$
2	$\mathbb{R}(2)$	${}^2\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	${}^2\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	${}^2\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$	$\mathbb{H}(32)$
5	${}^2\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	${}^2\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	${}^2\mathbb{H}(32)$
6	$\mathbb{H}(4)$	${}^2\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	${}^2\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	${}^2\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	${}^2\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	${}^2\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	${}^2\mathbb{R}(128)$	$\mathbb{R}(256)$

The famous *mod* 8 property of Clifford algebras allows to extend the table above for values $s, t > 8$.

The suffix " ^{2}" in the $s - t = 1$ *mod* 8 entries is introduced to take into account that, for such coupled values of s, t, a faithful representation of the Clifford algebra is obtained as a direct sum of its two inequivalent irreducible representations, see [?] for details.

Following [?] we have another possibility of understanding the connection between Clifford algebras and division algebras. We can simply state that a Clifford algebra is of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ type if its fundamental irreducible representation is realized in terms of matrices with entries in the corresponding division algebra. A constructive way of proving the above statement makes use of the two lifting algorithms [?], expressing the $Cl(s+1, t+1)$ and $Cl(t+2, s)$ Clifford irreps in terms of the Clifford irreps of $Cl(s, t)$ (given by the $s+t$ gamma matrices γ_i's). The $s+t+2$ gamma matrices Γ_j are given, in the two cases, by

$$\Gamma_i \equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ -\mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \tag{4.18}$$

and, respectively,

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1}_d & 0 \\ 0 & -\mathbf{1}_d \end{pmatrix} \tag{4.19}$$

The spinors carry a representation of the $Spin(s, t)$ spin group (see [?]), whose Lie algebra generators are given by the gamma matrices commutators. As a result, the division algebra structure of Gamma matrices extends to spinors. There is, however, for certain signatures of the space-time, a mismatch between division-algebra properties of the fundamental spinors and their associated Clifford algebras, see [?] and [?]. The mismatch is due to the existence of a Weyl-projection. We recall that, following [?], the fundamental spinors belong to the representation of the spin group admitting maximal division algebra structure. A table, presenting the division-algebra properties of spinors for $s, t \leq 8$, is here produced

$s\backslash t$	0	1	2	3	4	5	6	7	8
0		$\mathbb{R}^W$	$\mathbb{C}^W$	$\mathbb{H}$	$\mathbb{H}^W$	$\mathbb{H}(2)^W$	$\mathbb{C}(4)^W$	$\mathbb{R}(8)$	$\mathbb{R}(8)^W$
1	$\mathbb{R}$	$\mathbb{R}^W$	$\mathbb{R}(2)^W$	$\mathbb{C}(2)^W$	$\mathbb{H}(2)$	$\mathbb{H}(2)^W$	$\mathbb{H}(4)^W$	$\mathbb{C}(8)^W$	$\mathbb{R}(16)$
2	$\mathbb{C}^W$	$\mathbb{R}(2)$	$\mathbb{R}(2)^W$	$\mathbb{R}(4)^W$	$\mathbb{C}(4)^W$	$\mathbb{H}(4)$	$\mathbb{H}(4)^W$	$\mathbb{H}(8)^W$	$\mathbb{C}(16)^W$
3	$\mathbb{H}^W$	$\mathbb{C}(2)^W$	$\mathbb{R}(4)$	$\mathbb{R}(4)^W$	$\mathbb{R}(8)^W$	$\mathbb{C}(8)^W$	$\mathbb{H}(8)$	$\mathbb{H}(8)^W$	$\mathbb{H}(16)^W$
4	$\mathbb{H}^W$	$\mathbb{H}(2)^W$	$\mathbb{C}(4)^W$	$\mathbb{R}(8)$	$\mathbb{R}(8)^W$	$\mathbb{R}(16)^W$	$\mathbb{C}(16)^W$	$\mathbb{H}(16)$	$\mathbb{H}(16)^W$
5	$\mathbb{H}(2)$	$\mathbb{H}(2)^W$	$\mathbb{H}(4)^W$	$\mathbb{C}(8)^W$	$\mathbb{R}(16)$	$\mathbb{R}(16)^W$	$\mathbb{R}(32)^W$	$\mathbb{C}(32)^W$	$\mathbb{H}(32)$
6	$\mathbb{C}(4)^W$	$\mathbb{H}(4)$	$\mathbb{H}(4)^W$	$\mathbb{H}(8)^W$	$\mathbb{C}(16)^W$	$\mathbb{R}(32)$	$\mathbb{R}(32)^W$	$\mathbb{R}(64)^W$	$\mathbb{C}(64)^W$
7	$\mathbb{R}(8)^W$	$\mathbb{C}(8)^W$	$\mathbb{H}(8)$	$\mathbb{H}(8)^W$	$\mathbb{H}(16)^W$	$\mathbb{C}(32)^W$	$\mathbb{R}(64)$	$\mathbb{R}(64)^W$	$\mathbb{R}(128)^W$
8	$\mathbb{R}(8)^W$	$\mathbb{R}(16)^W$	$\mathbb{C}(16)^W$	$\mathbb{H}(16)$	$\mathbb{H}(16)^W$	$\mathbb{H}(32)^W$	$\mathbb{C}(64)^W$	$\mathbb{R}(128)$	$\mathbb{R}(128)^W$

The "W" denotes the presence of the Weyl projection. The numbers denote to the dimensionionality of the spinors. Just like the previous table, the division algebra properties of fundamental spinors for $s, t > 8$ are recovered from the *mod* 8 property of Clifford algebras.

The same type of analysis leading to the division-algebra properties of, respectively, Clifford algebras and fundamental spinors, can be repeated when investigating split-division algebra properties. The interesting case is that of split-quaternions ($\widetilde{\mathbb{H}}$) since, unlike the division-algebra case, split complex numbers and split quaternions are both represented in terms of 2×2 real matrices (complex numbers are represented by two 2×2 real matrices and quaternions by 4×4 real matrices). The basic example is provided by the $Cl(2,1)$ Clifford algebra, whose fundamental relation (4.17) can be realized in terms of the three split-quaternions of (2.10). The application of the lifting algorithms (4.18) and (4.19) allows to induce a split-quaternionic structure for the $Cl(s,t)$ Clifford algebras with

$$s = 2 + k \quad , \quad t = 1 + 8m + k, \quad for \quad m, k = 0, 1, 2, \ldots \tag{4.20}$$

and

$$s = 3 + 8m + k \quad , \quad t = 2 + k, \quad for \quad m, k = 0, 1, 2, \ldots \tag{4.21}$$

These Clifford algebras are the "oxidized forms" (according to [**?**]). In analogy with the construction in [**?**], reduced split-quaternionic Clifford algebras are obtained for $Cl(s-1,t)$ and $Cl(s-2,t)$, where s, t are either given by (4.20) or by (4.21).

At the end we obtain the table of split-quaternionic Clifford algebras given by

$s\backslash t$	1	2	3	4	5	6	7	8
1	$\widetilde{\mathbb{H}}$	0	0	0	0	0	0	0
2	$^2\widetilde{\mathbb{H}}$	$\widetilde{\mathbb{H}}(2)$	0	0	0	0	0	$\widetilde{\mathbb{H}}(16)$
3	$\widetilde{\mathbb{H}}(2)$	$^2\widetilde{\mathbb{H}}(2)$	$\widetilde{\mathbb{H}}(4)$	0	0	0	0	0
4	0	$\widetilde{\mathbb{H}}(4)$	$^2\widetilde{\mathbb{H}}(4)$	$\widetilde{\mathbb{H}}(8)$	0	0	0	0
5	0	0	$\widetilde{\mathbb{H}}(8)$	$^2\widetilde{\mathbb{H}}(8)$	$\widetilde{\mathbb{H}}(16)$	0	0	0
6	0	0	0	$\widetilde{\mathbb{H}}(16)$	$^2\widetilde{\mathbb{H}}(16)$	$\widetilde{\mathbb{H}}(32)$	0	0
7	0	0		0	$\widetilde{\mathbb{H}}(32)$	$^2\widetilde{\mathbb{H}}(32)$	$\widetilde{\mathbb{H}}(64)$	0
8	0	0	0	0	0	$\widetilde{\mathbb{H}}(64)$	$^2\widetilde{\mathbb{H}}(64)$	$\widetilde{\mathbb{H}}(128)$

Similarly, the split-quaternionic table for fundamental spinors is given by

$s\backslash t$	1	2	3	4	5	6	7	8
1	0	$\widetilde{\mathbb{H}}^W$	0	0	0	0	0	0
2	$\widetilde{\mathbb{H}}$	$\widetilde{\mathbb{H}}^W$	$\widetilde{\mathbb{H}}(2)^W$	0	0	0	0	0
3	0	$\widetilde{\mathbb{H}}(2)$	$\widetilde{\mathbb{H}}(2)^W$	$\widetilde{\mathbb{H}}(4)^W$	0	0	0	0
4	0	0	$\widetilde{\mathbb{H}}(4)$	$\widetilde{\mathbb{H}}(4)^W$	$\widetilde{\mathbb{H}}(8)^W$	0	0	0
5	0	0	0	$\widetilde{\mathbb{H}}(8)$	$\widetilde{\mathbb{H}}(8)^W$	$\widetilde{\mathbb{H}}(16)^W$	0	0
6	0	0	0	0	$\widetilde{\mathbb{H}}(16)$	$\widetilde{\mathbb{H}}(16)^W$	$\widetilde{\mathbb{H}}(32)^W$	0
7	0	0	0	0	0	$\widetilde{\mathbb{H}}(32)$	$\widetilde{\mathbb{H}}(32)^W$	$\widetilde{\mathbb{H}}(64)^W$
8	0	0	0	0	0	0	$\widetilde{\mathbb{H}}(64)$	$\widetilde{\mathbb{H}}(64)^W$

Both tables above can be extended for $s, t > 8$ due to the *mod* 8 property of Clifford algebras.

5 Split-division algebras and generalized supersymmetries

6 Conclusions

Appendix

We collect here for convenience, following [19] and [20], the main properties and definition of (split-)division algebras.

An algebra $\mathbb{A}$ over the reals ($\mathbb{R}$) is a composition algebra if it possesses a unit (denoted as $\mathbf{1}_{\mathbb{A}}$) and a non-degenerate quadratic form (norm) N satisfying

$$\begin{aligned} N(\mathbf{1}_{\mathbb{A}}) &= 1, \\ N(xy) &= N(x)N(y), \qquad \forall x, y \in \mathbb{A}. \end{aligned} \tag{A.1}$$

A composition algebra is alternative if the following left and right alternative properties are satisfied [21]

$$\begin{aligned}(x^2)y &= x(xy),\\ yx^2 &= (yx)x.\end{aligned} \tag{A.2}$$

A positive definite quadratic form is a mapping $N : \mathbb{A} \to \mathbb{R}^+$ s.t.

$$N(x) = 0 \quad \Leftrightarrow \quad x = 0. \tag{A.3}$$

A composition algebra with positive quadratic form is a division algebra, satisfying the property

$$xy = 0 \quad \Rightarrow \quad x = 0 \vee y = 0. \tag{A.4}$$

Due to the Hurwitz's theorem, the only division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

A $*$-algebra possesses a conjugation (i.e. an involutive automorphism $\mathbb{A} \to \mathbb{A}$) s.t., denoted as x^* the conjugate of $x \in \mathbb{A}$, we have

$$\begin{aligned}(x^*)^* &= x,\\ (xy)^* &= y^* x^*.\end{aligned} \tag{A.5}$$

The norm $N(x)$ of an element of a division algebra is expressed as

$$N(x) = xx^*. \tag{A.6}$$

Besides division algebras, we can introduce their split forms [19] as a new set of algebras. The split-division algebras are $*$-algebras with unit. The quadratic form N is no longer positive-definite and the property (A.4) is no longer valid. The algebras of split complex numbers, split quaternions and split octonions are respectively denoted as $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{H}}$ and $\widetilde{\mathbb{O}}$. The total number of inequivalent (split)-division algebras over $\mathbb{R}$ is 7 (the 4 division algebras and their 3 split forms above).

(Split)-division algebras find a unified description through the Cayley-Dickson doubling construction. Given an algebra $\mathbb{A}$ over $\mathbb{R}$, possessing a " $\cdot$ " multiplication, a " $*$" conjugation and a norm N, the Cayley-Dickson doubled algebra $\mathbb{A}^2$ over $\mathbb{R}$ is defined in terms of the operations in $\mathbb{A}$. Multiplication, conjugation and norm in $\mathbb{A}^2$ are respectively given by

i)	multiplication in $\mathbb{A}^2$:	$(x, y) \cdot (z, w) = (xz + \varepsilon w^* y, wx + yz^*)$,
ii)	conjugation in $\mathbb{A}^2$:	$(x, y)^* = (x^*, -y)$,
iii)	norm in $\mathbb{A}^2$:	$N(x, y) = N(x) - \varepsilon N(y)$.

The unit element $\mathbf{1}_{\mathbb{A}^2}$ of $\mathbb{A}^2$ is represented by $\mathbf{1}_{\mathbb{A}^2} = (\mathbf{1}_{\mathbb{A}}, 0)$.
In the above formulas ε is just a sign ($\varepsilon = \pm 1$).

It is convenient to denote the Cayley-Dickson's double of an algebra $\mathbb{A}$ by writing the ε sign on the right of the original algebra. For division algebras ε is always negative ($\varepsilon = -1$). We can therefore write

$\mathbb{C} = \mathbb{R}-$,

$\mathbb{H} = \mathbb{C}- = \mathbb{R} - -$,

$\mathbb{O} = \mathbb{H}- = \mathbb{C} - - = \mathbb{R} - --$.

The split division algebras are obtained by taking a positive ($\varepsilon = +1$) sign. We have

$\widetilde{\mathbb{C}} = \mathbb{R}+$,

$\widetilde{\mathbb{H}} = \mathbb{C}+ = \mathbb{R} - +$

$\widetilde{\mathbb{O}} = \mathbb{H}+ = \mathbb{C} - + = \mathbb{R} - -+$.

Other choices of the sign produce, at the end, isomorphic algebras. We can, e.g., also write $\widetilde{\mathbb{H}} = \mathbb{R} + +$, as well as $\widetilde{\mathbb{O}} = \mathbb{R} + ++$.

All (split-)division algebras over $\mathbb{R}$ are obtained by iteratively applying the Cayley-Dickson's construction starting from $\mathbb{R}$.

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