# Draft for the CBPF's 2006 School.

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**Abstract**

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# 1 Invariant off-shell actions. The $N = 4$ irreps. "Smart" methods

*i)* The $N = 4$ $(4,4)$ case. We have:

$$
\begin{array}{rcl}
Q_i(x, x_j; \psi, \psi_j) & = & (-\psi_i, \delta_{ij}\psi - \epsilon_{ijk}\psi_k; \dot{x}_i, -\delta_{ij}\dot{x} + \epsilon_{ijk}\dot{x}_k) \\
Q_4(x, x_j; \psi, \psi_j) & = & (\psi, \psi_j; \dot{x}, \dot{x}_j)
\end{array}
\tag{1.1}
$$

The most general invariant lagrangian $\mathcal{L}$ of dimension $d = 2$ constructed with the help of the enveloping representation is given by

$$
\begin{array}{rcl}
\mathcal{L} & = & \alpha(\vec{x})[\dot{x}^2 + \dot{x}_j^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] + \\
& & \partial_x\alpha[\psi\psi_j\dot{x}_j - \frac{1}{2}\epsilon_{ijk}\psi_i\psi_j\dot{x}_k] + \\
& & \partial_l\alpha[\psi_l\psi\dot{x} + \psi_l\psi_j\dot{x}_j + \frac{1}{2}\epsilon_{ljk}\psi_j\psi_k\dot{x} - \epsilon_{ljk}\psi_j\dot{x}_k\psi] - \\
& & -\Box\alpha\frac{1}{6}\epsilon_{ljk}\psi\psi_l\psi_k\psi_k
\end{array}
\tag{1.2}
$$

*ii)* The $N = 4$ $(3,4,1)$ case. We have:

$$
\begin{array}{rcl}
Q_i(x_j; \psi, \psi_j; g) & = & (\delta_{ij}\psi - \epsilon_{ijk}\psi_k; \dot{x}_i; -\delta_{ij}g + \epsilon_{ijk}\dot{x}_k; -\dot{\psi}_i) \\
Q_4(x_j; \psi, \psi_j; g_j) & = & (\psi_j; g, \dot{x}_j; \dot{\psi})
\end{array}
\tag{1.3}
$$

The most general invariant lagrangian $\mathcal{L}$ of dimension $d = 2$ constructed with the help of the enveloping representation is given by

$$
\begin{array}{rcl}
\mathcal{L} & = & \alpha(\vec{x})[\dot{x}_j^2 + g^2 - \psi\dot{\psi} - \psi_j\dot{\psi}_j] + \\
& & \partial_i\alpha[\epsilon_{ijk}(\psi\psi_j\dot{x}_k + \frac{1}{2}g\psi_j\psi_k) - g\psi\psi_i + \psi_i\psi_j\dot{x}_j] - \\
& & -\frac{\Box\alpha}{6}\epsilon_{ijk}\psi\psi_i\psi_j\psi_k
\end{array}
\tag{1.4}
$$

*iii)* The $N = 4$ $(2,4,2)$ case. We have:

$$
\begin{array}{rcl}
Q_1(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) & = & (\psi_0, \psi_3; \dot{x}, -g, h, -\dot{y}; -\dot{\psi}_1, \dot{\psi}_2) \\
Q_2(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) & = & (\psi_3, \psi_0; \dot{y}, -h, -g, \dot{x}; -\dot{\psi}_2, -\dot{\psi}_1) \\
Q_3(x, y; \psi_0, \psi_1, \psi_2, \psi_3; g, h) & = & (-\psi_2, \psi_1; h, \dot{y} - \dot{x}, -g; -\dot{\psi}_3, \dot{\psi}_0) \\
Q_4(x, y; \psi_0, \psi_1\psi_2, \psi_3; g, h) & = & (\psi_1, \psi_2; g, \dot{x}, \dot{y}, h; \dot{\psi}_0, \dot{\psi}_3)
\end{array}
\tag{1.5}
$$

The most general invariant lagrangian $\mathcal{L}$ of dimension $d=2$ constructed with the help of the enveloping representation is given by

$$\begin{aligned}
\mathcal{L} &= \alpha(x,y)[\dot{x}^2+\dot{y}^2+g^2+h^2-\psi\dot{\psi}-\psi_j\dot{\psi}_j]+\\
&\quad \partial_x\alpha[\dot{y}(\psi_1\psi_2-\psi_0\psi_3)+g(\psi_2\psi_3-\psi_0\psi_1)+h(\psi_1\psi_3+\psi_0\psi_2)]+\\
&\quad \partial_y\alpha[-\dot{x}(\psi_1\psi_2-\psi_0\psi_3)-g(\psi_1\psi_3+\psi_0\psi_2)+h(\psi_2\psi_3-\psi_0\psi_1)]-\\
&\quad -\Box\alpha\psi_0\psi_1\psi_2\psi_3
\end{aligned} \tag{1.6}$$

*iv*) The $N=4$ $(1,4,3)$ case. We have:

$$\begin{aligned}
Q_i(x;\psi,\psi_j,g_j) &= (-\psi_i;g_i,-\delta_{ij}\dot{x}+\epsilon_{ijk}g_k;\delta_{ij}\dot{\psi}-\epsilon_{ijk}\dot{\psi}_k),\\
Q_4(x;\psi,\psi_j;g_j) &= (\psi;\dot{x},g_j;\dot{\psi}_j)
\end{aligned} \tag{1.7}$$

The most general invariant lagrangian $\mathcal{L}$ of dimension $d=2$ constructed with the help of the enveloping representation is given by

$$\begin{aligned}
\mathcal{L} &= \alpha(x)[\dot{x}^2-\psi\dot{\psi}-\psi_i\dot{\psi}_i+{g_i}^2]+\\
&\quad \alpha'(x)[\psi g_i\psi_i-\frac{1}{2}\epsilon_{ijk}g_i\psi_j\psi_k]-\frac{\alpha''(x)}{6}[\epsilon_{ijk}\psi\psi_i\psi_j\psi_k]
\end{aligned} \tag{1.8}$$

# 2 Invariant off-shell actions. The $N=8$ irreps.

*i*) The $(1,8,7)$ case has been discussed in the [1] paper. It is produced with the octonionic covariantization from the $(1,4,3)$ case. The general function $\alpha(x)$ is constrained to be linear, i.e. $\alpha(x)=ax+b$, where $a,b$ are constants.

*ii*) The $(7,8,1)$ case has been analyzed within the octonionic covariantization procedure, starting from the $(3,4,1)$ case. It implies a trivial (constant $\alpha$) kinetic energy invariant.

*iii)* The $(8,8)$ case can be analyzed within the octonionic covariantization procedure starting from the $(4,4)$ case. The result will be produced next.

*iv)* Some of the remaining cases could be analyzed through a split octonionic covariantization procedure?

# 3 Invariant off-shell actions. The $N=9$ irreps.

This case involve 16 bosons and 16 fermions. It is hopefully that an octonionic covariantization procedure could produce results for some of the multiplets. The most interesting one, see later, seems to be the $(9,16,7)$, see [2].

# 4 Use of brute-force method to construct off-shell invariant actions.

Steps: take the most general lagrangian $\mathcal{L}_i$ term of dimension $d = i$ and check under which condition the action $\mathcal{S}_i = \int dt \mathcal{L}_i$ is invariant under the whole set of supersymmetry transformations.

For our purposes the most interesting invariant actions are

*i)* $\mathcal{L}_2$, the "kinetic term",

*ii)* $\mathcal{L}_1$, the "potential term",

*iii)* $\mathcal{L}_0$, the "quartic term".

The invariant actions $\mathcal{L}_d$ of mass-dimension $d > 2$ admit higher-order derivatives. They can be associated either with Born-Infeld non-linear lagrangians, or non-local perturbative corrections in the $\hbar$ expansion beyond the tree level and so on.

# 5 Dimensional analysis and dimensional reduction.

There are three fundamental constants in physics:

*i)* the velocity of light $c$,

*ii)*the Planck's constant $\hbar$,

*iii)* the Newton's gravitational constant $G_N$.

$c$ is the realm of special relativity. Its constancy implies that we can trade length and time. In natural units we can set $c = 1$ and use either centimeters or seconds to measure both space and time.

$\hbar$ is the realm of quantum mechanics. Setting $\hbar = 1$ we can interchange the unit of energy (or mass) with the unit of time (the second).

Relativistic quantum mechanics implies that in physics there is only one type of dimension (call it length, time, or energy) which can be measured, interchangeably, in centimeters, seconds or GeVs.

The gravitational constant $G_N$ is a dimensional constant which corresponds, in natural units, to

*i)* $10^{-33}$ centimeters, or

*ii)* $10^{-43}$ seconds, or

*iii)* $10^{19}$ GeV.

These numbers have been produced by playing with the whole set of the three fundamental constants $c$, $\hbar$, $G_N$. They are therefore related with the (unknown) full theory of relativistic quantum gravity. They represent the scale of magnitude where relativistic quantum gravity should be appreciated. We have, respectively,

*i)* at probing distances of the order $10^{-33}$ cm (the atomic distance is of the order of $10^{-13}$ cm),

*ii)* at $10^{-43}$ seconds "after" the Big Bang, whatever this means (the present age of the universe is of the order of 10 billion years),

*iii)* at "very" hypotethical (and totally unphysical) colliders working at $10^{19}$ Gev (the energy we can presently achieve are of the order of $10^2 - 10^3$ Gev). Even if some "smart" scenarios (involving branes) can be produced to lower the scale of relativistic quantum gravity at the order of the Tev, making them palatable, being on the range of foreseeable future experiments, we should not forget that the huge, physically inaccessible, value of $10^{19}$ Gev is the natural order of magnitude of relativistic quantum gravity. It comes to no surprise then that the quantum unification of all interactions comes at a scale which makes physical predictions at the experimentable scale of length if not totally impossible, at least extremely hard. The golden rule to find a quantum unifying theory of all interactions is replaced from adherence to experiments to mathematical consistency conditions (see e.g. the role of the anomalies in assigning the charges, separately for leptons and quarks and for each three families). Coming back to the three fundaental numbers above, they will be referred to, respectively, as

*i*) the Planck's length $l_P = 10^{-33}$ cm,

*ii*) the Planck's time $t_P = 10^{-43}$ sec,

*iii*) the Planck's energy (mass) $M_P = 10^{19}$ GeV.

In the following, we will fix the *mass* as the standard unit of reference. In natural units, $G_N$ is dimensionful, with mass dimension $[G_n] = -2$.

# 6 Dimensional analysis of dimensionally reduced theories

We make here dimensional analysis of the following theories:

*i*) the free particle in one (time) dimension ($D = 1$) and,

for the ordinary Minkowski space-time ($D = 4$) the

*iia*) the scalar boson theory (with quartic potential $\frac{\lambda}{4!}\phi^4$),

*iib*) the Yang-Mills theory and,

*iic*) the gravity theory (expressed in the vierbein formalism).

We further make the dimensional analysis of the above three theories when dimensionally reduced (*à la Scherk*) to a one (time) dimensional $D = 1$ quantum mechanical system.

We further repeat the dimensional analysis for the supersymmetric version of these theories and end up with the dimensional analysis of the

*iii*) string theory.

*Case i*) - *the $D = 1$ free particle* It is described by a dimensionless action $\mathcal{S}$ given by

$$\mathcal{S} = \frac{1}{m}\int dt\dot{\varphi}^2 \tag{6.9}$$

The dot denotes, as usual, the time derivative. Since the time $t$ has the inverse of a mass-dimension ($[t] = -1$), assuming $\varphi$ being dimensionless ($[\varphi] = 0$), an overall constant (written as $\frac{1}{m}$) of mass dimension $-1$ has to be inserted to make $\mathcal{S}$ adimensional. Summarizing we have, for this $D = 1$ model

$$\begin{aligned} [t]_{D=1} &= -1, \\ [\frac{\partial}{\partial t}]_{D=1} &= 1, \end{aligned}$$

$$
\begin{aligned}
[\varphi]_{D=1} &= 0, \\
[m]_{D=1} &= 0, \\
[\mathcal{S}]_{D=1} &= 0.
\end{aligned} \tag{6.10}
$$

The suffix $D = 1$ has been added for later convenience, rmembering that the dmensional analysis corresponds to the one-dimensional model (this will be useful later on, when discussing the dimensional reduction of higher dimensional theories).

*Case iia) -the $D = 4$ scalar boson theory* The action can be given by

$$
\mathcal{S} = \int d^4x \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} M^2 \Phi^2 - \frac{1}{4!} \lambda \Phi^4 \right), \tag{6.11}
$$

An adimensional action $\mathcal{S}$ is obtained by setting, in mass dimension

$$
\begin{aligned}
[\Phi]_{D=4} &= 1, \\
[\partial_\mu]_{D=4} &= 1, \\
[M]_{D=4} &= 1, \\
[\lambda]_{D=4} &= 0.
\end{aligned} \tag{6.12}
$$

*Case iib) -the $D = 4$ pure QED or Yang-Mills theories.*

The gauge-invariant action is given by

$$
\mathcal{S} = \frac{1}{e^2} \int d^4x Tr \left( F_{\mu\nu} F^{\mu\nu} \right), \tag{6.13}
$$

where the antisymmetric stress-energy tensor $F_{\mu\nu}$ is given by

$$
F_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu] \tag{6.14}
$$

with $\mathcal{D}_\mu$ the covariant derivative, expressed in terms of the gauge connection $A_\mu$

$$
\mathcal{D}_\mu = \partial_\mu - e A_\mu \tag{6.15}
$$

$e$ is the charge (the electric charge for QED). The action is adimensional, provided that

$$
\begin{aligned}
[A_\mu]_{D=4} &= 1, \\
[F_{\mu\nu}]_{D=4} &= 2, \\
[e]_{D=4} &= 0.
\end{aligned} \tag{6.16}
$$

*iic) -The pure gravity case.*

The action is constructed, see [?] for details, in terms of the determinant $\mathcal{E}$ of the vierbein $e_\mu{}^a$ and the curvature scalar $\mathcal{R}$. It is given by

$$
\mathcal{S} = \frac{-6}{8\pi G_N} \int d^4x \mathcal{E} \mathcal{R} \tag{6.17}
$$

The overall constant (essentially the inverse of the gravitational constant $G_N$) is now dimensional ($[G_N]_{D=4} = -2$). The adimensional action is recovered by setting

$$
\begin{aligned}
[e_\mu{}^a]_{D=4} &= 0, \\
[\mathcal{R}]_{D=4} &= 2.
\end{aligned} \tag{6.18}
$$

Let us now discuss the dimensional reduction from $D = 4 \Rightarrow D = 1$. Let us suppose that the three space dimensions belong to some compact manifold $\mathcal{M}$ (e.g. the three-sphere $S^3$) and let us freeze the dependence of the fields on the space-dimensions (the application of the time derivative $\partial_0$ leads to non-vanishing results, while the application of the space-derivatives $\partial_i$, for $i = 1, 2, 3$ gives zero). Our space-time is now given by $\mathbb{R} \times \mathcal{M}$. We get that the integration over the three space variables contributes just an overall factor, the volume of the three-dimensional manifold $\mathcal{M}$. Therefore

$$\int d^4x \quad \equiv \quad Vol_{\mathcal{M}} \cdot \int dt \tag{6.19}$$

Since

$$[Vol_{\mathcal{M}}]_{D=4} \quad = \quad -3 \tag{6.20}$$

we can express $Vol \equiv \frac{1}{m^3}$, where $m$ is a mass-term. A factor $\frac{1}{m}$ contributes as the overall factor in the one-dimensional theory, while the remaining part $\frac{1}{m^2}$ can be used to rescale the fields. We have, e.g., for the dimensional reduction of the scalar boson theory that

$$\varphi_{D=1} \quad \equiv \quad \frac{1}{m}\phi_{D=4} \tag{6.21}$$

The dimensional reduction of the scalar boson theory *ii a* is given by

$$\mathcal{S} \quad = \quad \frac{1}{m}\int dt \left(\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}M^2\varphi^2 + \lambda_{D=1}\frac{1}{4!}\varphi^4\right) \tag{6.22}$$

where we have

$$\begin{aligned}
[\varphi]_{D=1} &= 0, \\
[M]_{D=1} &= 1, \\
[\lambda_1]_{D=1} &= 2
\end{aligned} \tag{6.23}$$

The $D = 1$ coupling constant $\lambda_1$ is related to the $D = 4$ adimensional coupling constant $\lambda$ by the relation

$$\lambda_1 \quad = \quad \lambda m^2. \tag{6.24}$$

We proceed in a similar way in the case of the Yang-Mills theory. We can rescale the $D = 4$ Yang-Mills fields $A_\mu$ to the $D = 1$ fields $B_\mu = \frac{1}{m}A_\mu$. The $D = 1$ charge $e$ is rescaled to $e_1 = em$. We have, symbolically, for the dimensional reduced action, a sum of terms of the type

$$\mathcal{S} \quad = \quad \frac{1}{m}\int dt \left(\dot{B}^2 + e_1\dot{B}B^2 + {e_1}^2B^4\right) \tag{6.25}$$

where

$$\begin{aligned}
[B]_{D=1} &= 0, \\
[e_1]_{D=1} &= 1.
\end{aligned} \tag{6.26}$$

The situation is different for what concerns the gravity theory. In that case, the overall factor $Vol_{\mathcal{M}}/G_N$ produces the dimensionally correct $\frac{1}{m}$ overall factor of the one-dimensional theory. This implies that we do not need to rescale the dimensionality of the vierbein ${e_\mu}^a$ and of the curvature. Summarizing we have the following results

| | | | | | |
|---|---|---|---|---|---|
| scalar boson | $\Phi$ | : | $[\Phi]_{D=4}=1$ | $\Rightarrow$ | $[\Phi]_{D=1}=0$ |
| gauge connection | $A_\mu$ | : | $[A_\mu]_{D=4}=1$ | $\Rightarrow$ | $[A_\mu]_{D=1}=0$ |
| vierbein | ${e_\mu}^a$ | : | $[{e_\mu}^a]_{D=4}=1$ | $\Rightarrow$ | $[{e_\mu}^a]_{D=1}=0$ |
| electric charge | $e$ | : | $[e]_{D=4}=0$ | $\Rightarrow$ | $[e]_{D=1}=1$ |

(6.27)

Let us now discuss the $N=1$ supersymmetric version of the three $D=4$ theories above. The chiral multiplet, described in [**?**], in terms of the chiral superfields $\Phi$, $\overline{\Phi}$ admits the following field

| | | |
|---|---|---|
| chiral multiplet | : | $\Phi$, $\overline{\Phi}$ |
| fields content | : | $(2,4,2)$ |
| $D=4$ dimensionality | : | $[1,\frac{3}{2},2]_{D=4}$ |
| $D=1$ dimensionality | : | $[0,\frac{1}{2},1]_{D=1}$ |
| vector multiplet | : | $V=V^\dagger$ |
| fields content | : | $(1,4,6,4,1)$ |
| $D=4$ dimensionality | : | $[0,\frac{1}{2},1,\frac{3}{2},2]_{D=4}$ |
| $D=1$ dimensionality | : | $[-1,-\frac{1}{2},0,\frac{1}{2},1]_{D=1}$ |
| vector multiplet | : | $V$ in the WZ gauge |
| fields content | : | $(3,4,1)$ |
| $D=4$ dimensionality | : | $[1,\frac{3}{2},2]_{D=4}$ |
| $D=1$ dimensionality | : | $[0,\frac{1}{2},1]_{D=1}$ |
| supergravity multiplet | : | ${e_\mu}^a$, ${\psi_\mu}^\alpha$ |
| fields content | : | $(16,16)$ |
| $D=4$ dimensionality | : | $[0,\frac{1}{2}]_{D=4}$ |
| $D=1$ dimensionality | : | $[0,\frac{1}{2}]_{D=1}$ |
| gauged sugra multiplet | : | ${e_\mu}^a$, ${\psi_\mu}^\alpha$, $b^i$ |
| fields content | : | $(6,12,6)$ |
| $D=4$ dimensionality | : | $[0,\frac{1}{2},1]_{D=4}$ |
| $D=1$ dimensionality | : | $[0,\frac{1}{2},1]_{D=1}$ |

(6.28)

*Case iii) - the bosonic string.*

The action is given by

$$\mathcal{S} \;=\; -\frac{1}{2\pi\alpha'}\int d\tau d\sigma \left((\dot{X}^\mu X'_\mu)^2-(\dot{X}_\mu\dot{X}^\mu)({X_\nu}'X^{\nu\prime})\right) \qquad (6.29)$$

There are two space-times here: the two-dimensional world-sheet and the target space-time. Therefore there are two dimensional quantities whether we look at the object from the two-dimensional or the target manifold point of view. We can assign $[\cdot]_{ws}$ and $[\cdot]_{trg}$ dimensions. We have

$$
\begin{array}{rcl}
[\tau]_{ws} = [\sigma]_w s & = & -1, \\
{[X_\mu]_{ws}} & = & 0, \\
{[\alpha']_{ws}} & = & 0
\end{array} \qquad (6.30)
$$

and

$$
\begin{array}{rcl}
[\tau]_{trg} = [\sigma]_{trg} & = & 0, \\
{[X_\mu]_{trg}} & = & -1, \\
{[\alpha']_{trg}} & = & -2.
\end{array} \qquad (6.31)
$$

# 7 The Fano's plane (some fun with Fano)

The seven dimensional Fano's plane $FP$ is the simplest example of finite projective geometry. All points are equivalent. 7 points (labeled $1, 2, \ldots, 7$) and 7 lines consisting of 3 points (3 lines intersect at each point). For the octonions we need to introduce arrows on the line. We can say that the corresponding Fano's plane is oriented. Some questions: how many inequivalent ways do exist of embedding $\mathbb{R}_7$ in $FP$? Call $x_1, \ldots, x_7$ the oriented sets of coordinates of $\mathbb{R}_7$. Without loss of generality we can fix $x_1 \equiv 1$ and $x_2 \equiv 2$. There are 5 inequivaent choices to put one of the remaining points in 3. Once set, let us call $y_1, y_2, y_3, y_4$ the rordered set of reaining points. Without loss of generality we can set $y_1 \equiv 4$. Now there are 6 inequivalent choices to set $y_2, y_3, y_4$ in the Fano's position 5, 6 and 7.

Therefore there are 30 inequivalent configurations for the arrangement of 7 ordered points in the Fano's plane.

The total number of permutations of seven points is 7!. The number of permutations resulting in an equivalent arrangement of points in $FP$ is given by $7!/30 = 168$. This number corresponds to the order of the finite group of automorphisms of the Fano's plane, the projective group $PSL(2, 7)$ of 168 elements. It is a simple non-abelian finite group (the unique simple group of order 168). It is the second largest nonabelian finite group after the alternating group $A_5$ of order 60.

The next question is: how many inequivalent ways do we have to embed $\mathbb{R}_8$ in the octonionic space $\mathbb{O}$. The latter is given by a single element (corresponding to the identity) plus the seven imaginary octonions arranged in an ordered Fano's plane (the lines now have arrows). We get the following numbers: 8 ways of setting the $x$'s in the identity. Call $y_1, \ldots, y_7$ the remaining points. Without loss of generality we can set $y_1$ and $y_2$ in the positions 1 and 2 (in 2 inequivalent ways, due to the arrow linking 1 and 2. There remains, as before 5 choices to put the another coordinate in 3. The remaining 4 points , labeled now as $z_1, \ldots, z_4$ have to be put in position $4, 5, 6$ and 7. Take $z_1$, one can easily verifies that setting it in the above positions leads to inequivalent configurations. We have therefore 4! possibilities to arrange the remaining 4 points. The total number of inequivalent configurations is therefore given by $8 \times 2 \times 5 \times 4! = 1920$. The finite group of transformations leaving invariant $\mathbb{O}$ is given by $8!/1920 = 21$ elements. It corresponds to the group $\mathbf{Z}_3 \otimes \mathbf{Z}_7$. It is given by the seven permutations of the position of the point 1 and the even cyclic permutations of the three lines converging at each point.

# 8 Connection between Grassmann algebra, Clifford algebra and $N$-extended $D = 1$ supersymmetry algebra

The Grassmann algebra is the enveloping algebra generated by the $N$ generators $\theta_a$ ($a = 1, \ldots, N$) satisfying the relation

$$\theta_a\theta_b + \theta_b\theta_a = 0. \tag{8.32}$$

The Clifford algebra is the enveloping algebra generated by the $N$ generators $\gamma_i$ ($i = 1, \ldots, N$) satisfying the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}\mathbf{1}. \tag{8.33}$$

where $\eta_{ij}$ is a diagonal matrix and $\mathbf{1}$ is the identity operator.

In order to "promote" the basic relation of the Clifford algebra as the constituent relation of a (super)Lie algebra $\mathcal{G}$ (with a mapping $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ we have to interpret the $N$ $\gamma_i$ generators on the l.h.s. as the odd elements of a super-Lie algebra (we will call them $Q_i$'s) , and we have to add the identity $\mathbf{1}$ as an even element of the superalgebra corresponding at a central extension $z$ (since $\mathbf{1}$ commutes with the $\gamma_i$'s. We are led to a superalgebra (the $N$-extended (pseudo)susy algebra) with total number of $N$ odd elements $Q_i$ and a single even element, the central extension $z$. The $N$-susy superalgebra is defined by the relations

$$\begin{aligned} \{Q_i, Q_j\} &= \eta_{ij}z, \\ [Q_i, z] &= 0 \end{aligned} \tag{8.34}$$

If $\eta_{ij} \equiv \delta_{ij}$ the $N$-extended pseudo superalgebra is called the one-dimensional $N$ extended supersymmetry algebra (from now on, for short, $N$ susy).

In physics, the central extension $z$ is denoted with $H$ and called the hamiltonian.

# 9 some miscellanea's facts about invariants and longer length multiplets

The $N = 16$-susy algebra admits irreps of order $128 + 128$ (bosons+fermions). It is obtained from $Cl(0, 15)$. The latter can be reconstructed through the structure constants of the sedenions.

Our classificatory result in [1] implies as a corollary that, for sufficiently large values of $N$ ($N \geq 10$ there exists irreducible multiplets which can accommodate fields of dimensions $0, \frac{1}{2}, 1, \frac{3}{2}, 2$. In ordinary invariant off-shell lagrangians (whose terms have dimensions 0, 1 and 2, the longest fields of dimension $\frac{3}{2}$ and 2 can only enter as lagrange multipliers.

The $(3,4,1)$ "vector" multiplet of $N=4$ admits the standard (constant) kinetic term corresponding to the dimensional reduction of the pure super-QED action. There exists in one dimension a 1-dimensional term, which can enter an invariant lagrangian, which is only $N=1$ invariant. It is given by

$$\mathcal{L}_{d=1} = \psi\psi_j x_j - \frac{1}{2} g {x_j}^2. \tag{9.35}$$

What is the situation for the $N=8$ $(5,8,3)$ vector multiplet and the$N=9$ $(9,16,7)$ vector multiplet? For $(5,8,3)$ we expect, by dimensional reduction considerations, that it admits an $N8$ invariant term of dimension 1 (trilinear in the fields and proportional to the charge $e$) and an $N=8$ invariant term of dimension 0 (quartilinear in the fields and proportional to $e^2$).

# 10 The BRST complex of the N-susy

Following [?] we can specialize the formulas to the $N$-susy algebra. We get

# 11 The oxidation program

# 12 Dictionary on $N$-extended susy irreps

# References


[1] Z. Kuznetsova, M. Rojas and F. Toppan, hep-th/0511274. To appear in JHEP.

[2] N. Berkovits, Phys. Lett. **B 318** (1993) 104. In hep-th/9308128.