CHERN-SIMONS (SUPER-)GRAVITIES
(v.2.3)

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Abstract: These lectures are mostly a revised version of the lecture notes prepared for the *Villa de Leyva Summer School 2001* and which were published in [1], which was in turn an updated version of the lectures at the *La Hechicera School 1999* in Mérida (Venezuela). They are intended as a broad introduction to Chern-Simons gravity and supergravity. The motivation for these theories lies in the desire to have a gauge invariant action –in the sense of fiber bundles– in more than three dimensions, which could provide a firm ground for constructing a quantum theory of the gravitational field. The case of Chern-Simons gravity and its supersymmetric extension for all odd D is presented. No analogous construction is available in even dimensions.

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PART ONE

MEN WANTED
FOR HAZARDOUS JOURNEY.
LOW WAGES, BITTER COLD,
LONG MONTHS OF COMPLETE
DARKNESS. CONSTANT DANGER,
SAFE RETURN DOUBTFUL.
HONOUR AND RECOGNITION
IN CASE OF SUCCESS.

ERNEST SHACKLETON[2]

1 The Quantum Gravity Puzzle

The construction of the action principle for general relativity is reviewed from a modern perspective. The analysis avoids as much as possible to discuss coordinates and invariance under coordinate transformations. Instead, it uses the assumption that the spacetime is a smooth manifold and hence it is endowed with a tangent space which is isomorphic to the Minkowski spacetime. In this description of the spacetime geometry, the metric and affine properties are represented by independent fields, an idea that goes back to the works of Cartan and Palatini. This leads naturally to a formulation of gravity in terms of two independent 1-form fields: the vielbein, e^a, and the spin connection $\omega^a_{\ b}$. The construction is valid for a theory of gravity in any number of dimensions and makes the similarities and contrasts between gravity and a gauge theory explicit.

1.1 Renormalizability and the triumph of gauge theory

1.1.1 Quantum Field Theory

The Standard Model of high energy physics is a remarkably successful, enormously precise and predictive theory for particle interactions. With this model, three of the four forces of nature (electromagnetism, weak, and strong interactions) are explained and accurately described. The dynamical structure of the model is a Yang-Mills action, built on the assumption that nature should be invariant under a group of transformations acting independently at each point of spacetime: a local, or "gauge", symmetry. This symmetry fixes almost uniquely the types of couplings among all the fields that describe both the basic constituents of matter, and the carriers of their interactions.

A most important feature of the standard model is the way in which it avoids inconsistencies: renormalizability, absence of anomalies, etc. In fact, it is this capability which makes it a believable tool and at the same time, it is the prime criterion for its construction.

The remarkable thing is that renormalizability and lack of anomalies are tremendously restrictive conditions, so that a very limited number of possible actions pass the test, which is reassuring. The opposite would be embarrassing: having a large a number of physically sensible theories would prompt the question, how come why we don't see the others?

Although not all gauge invariant theories are guaranteed to be renormalizable, the only renormalizable theories that describe our universe are gauge theories. That this is so is an unexpected bonus of the gauge principle since gauge invariance was not introduced to cure the renormalizability problem but rather as a systematic way to bring about interactions that would respect a given symmetry.

Thus, gauge invariance seems to be a crucial ingredient in the construction of physically testable (renormalizable) theories. Symmetry principles, then, are not only useful in constructing the right (classical) action functionals, but they are often sufficient to ensure the viability of a quantum theory built from a given classical action. An intuitive way to understand the usefulness of gauge invariance in a quantum setting is that the gauge symmetry does not depend on the field configurations. Now, if this symmetry relates the divergences appearing in the scattering amplitudes in such a way that they can be absorbed by a redefinition of the parameters in the action at a certain order in the semiclassical (loop) expansion, it should do the same at all orders, as the symmetry is not spoiled by quantum corrections.

In contrast with this, are symmetries which are only realized if the field obey the classical equations of motion, something that is often referred to as an *on shell symmetry*. In general, on shell symmetries are not respected by quantum mechanics.

The underlying structure of the gauge principle is mathematically captured through the concept of **fiber bundle**, which is a systematic way to implement a group acting on a set of fields that carry a particular representation of the group. For a discussion of the physical applications, see [3].

1.1.2 Enter Gravity

The fourth interaction of nature, the gravitational attraction, has stubbornly resisted quantization. Attempts to set up a semiclassical or perturbative expansion of the theory have systematically failed and the progress in this direction has remained rather formal, and after more than 70 years of efforts, no consistent quantum theory of gravity in four dimensions is known[2].

The straightforward perturbative approach to quantum gravity was explored by many authors, starting with Feynman [5] and it was soon realized that the perturbative expansion is nonrenormalizable (see, e.g., [6]). One way to see this

[2]In three spacetime dimensions, the analogous of Einstein-Hilbert gravity is quantizable, although it does not possess local, propagating degrees of freedom [4].

is that if one splits the gravitational lagrangian as a kinetic term plus interaction, the coupling constant for the interaction among gravitons is Newton's constant G, which has dimensions of $mass^{-2}$. This means that the diagrams with larger number of vertices will require higher powers of momentum in the numerators compensate the dimensions, and this in turn means that one can expect ultraviolet divergences of all powers to be present in the perturbative expansion. The lesson one can learn from this frustrating exercise is that General Relativity in its most naive interpretation as an ordinary field theory for the metric is at best an effective theory. For an interesting discussion of how one can live with an effective theory of gravity, see the recent review by Burgess [7].

The situation with quantum gravity is particularly irritating because we have been led to think that the gravitational attraction is a fundamental interaction. The dynamical equations of complex systems, like fluids and dispersive media, are not amenable to a variational description, they are not truly fundamental and therefore one should not expect to have a quantum theory for them. The gravitational field on the other hand, is governed by the Einstein equations, which in turn can be derived from an action principle. This is why we could expect, in principle, to define a path integral for the gravitational field, even if calculating with it could be a very nontrivial issue [8].

General Relativity seems to be the only consistent framework that describes gravitational phenomena, compatible with the principle that physics should be insensitive to the state of motion of the observer. This principle is formally translated as invariance under general coordinate transformations, or general covariance. This invariance is a local symmetry, analogous to the gauge invariance of the other three forces, and one could be tempted to view it as a gauge symmetry, with interesting consequences. Unfortunately, general relativity doesn't qualify as a gauge theory, except for a remarkable accident in three spacetime dimensions.

One of the differences between a coordinate transformation and a proper gauge transformation is most manifest by the way in which they act on the fields: coordinate transformations change the arguments as well as the field components, whereas a gauge transformation leaves the arguments unchanged. But this is not a very serious obstruction and one can find the right combination of fields so that a change of frame does not change the coordinates. This can be done adopting the tangent space representation and that is what we will do here. A more serious problem is to prove that the action for gravity is invariant under the group of local translations, which is the tangent space representation for diffeomorphism invariance. It turns out that the gravitational action changes by a term which vanishes if the field equations hold, but not otherwise, which means that this is at best an *on-shell symmetry*.

In these lectures we attempt to shed some light on this issue, and will show how to generalize the three dimensional accident to higher dimensions.

1.2 Minimal Couplings and Connections

Gauge symmetry fixes the form in which matter fields couple to the carriers of interactions. In electrodynamics, for example, the ordinary derivative in the kinetic term for the matter fields, ∂_μ, is replaced by the covariant derivative,

$$\nabla_\mu = \partial_\mu + A_\mu, \tag{1}$$

and this accounts for all electromagnetic interactions with matter.

Thus, the gauge coupling provides a unique way to describe interactions with charged fields. At the same time, it doesn't require the introduction of dimensionful constants in the action, which is a welcome feature in perturbation theory since the expansion is likely to be well behaved. Also, gauge symmetry severely restricts the type of counterterms that can be added to the action, as there are very few gauge invariant expressions in a given number of spacetime dimensions. Thus, if the lagrangian contains all possible terms allowed by the symmetry, perturbative corrections could only lead to rescalings of the coefficients in front of each term in the lagrangian. These rescalings, in turn can always be absorbed in a redefinition of the parameters of the action, which is why the renormalization procedure works in gauge theories and is the key to their internal consistency.

The "vector potential" A_μ is a connection 1-form, which means that, under a gauge transformation,

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}(\mathbf{x})' = \mathbf{U}(x)\mathbf{A}(\mathbf{x})\mathbf{U}(x)^{-1} + \mathbf{U}(x)d\mathbf{U}^{-1}(x), \tag{2}$$

where $\mathbf{U}(x)$ represents a position-dependent group element. The value of $\mathbf{A}$ depends on the choice of gauge $\mathbf{U}(x)$ and it can even be made to vanish at a given point by an appropriate choice of $\mathbf{U}(x)$. The combination ∇_μ is the covariant derivative, a differential operator that, unlike the ordinary derivative and $\mathbf{A}$ itself, transforms homogeneously under the action of the gauge group,

$$\nabla_\mu \to \nabla'_\mu = \mathbf{U}(x)\nabla_\mu. \tag{3}$$

The connection is in general a matrix-valued object. For instance, in the case of nonabelian gauge theories, ∇_μ is an operator 1-form,

$$\begin{aligned} \nabla &= d + \mathbf{A} \\ &= dx^\mu(\partial_\mu + \mathbf{A}_\mu). \end{aligned} \tag{4}$$

Acting on a function $\phi(x)$, which is in a vector representation of the gauge group ($\phi(x) \to \phi'(x) = \mathbf{U}(x) \cdot \phi(x)$), the covariant derivative reads

$$\nabla\phi = d\phi + \mathbf{A} \wedge \phi. \tag{5}$$

The covariant derivative operator ∇ has a remarkable property: its square is not a differential operator but a multiplicative one, as can be seen from (5)

$$\begin{aligned} \nabla\nabla\phi &= d(\mathbf{A}\phi) + \mathbf{A}d\phi + \mathbf{A}\wedge\mathbf{A}\phi \\ &= (d\mathbf{A} + \mathbf{A}\wedge\mathbf{A})\phi \\ &= \mathbf{F}\phi \end{aligned} \tag{6}$$

The combination $\mathbf{F} = d\mathbf{A} + \mathbf{A}\wedge\mathbf{A}$ is the field strength of the nonabelian interaction. This generalizes the electric and magnetic fields of electromagnetism and it indicates the presence of energy.

One can see now why the gauge principle is such a powerful idea in physics: the covariant derivative of a field, $\nabla\phi$, defines the coupling between ϕ and the gauge potential $\mathbf{A}$ in a unique way. Furthermore, $\mathbf{A}$ has a uniquely defined field strength $\mathbf{F}$, which in turn defines the dynamical properties of the gauge field. In 1954, Robert Mills and Chen-Nin Yang grasped the beauty and the power of this idea and constructed what has been since known as the nonabelian Yang-Mills theory [9].

On a curved manifold, there is another operator analogous to ∇, also called the covariant derivative in differential geometry,

$$\begin{aligned} \mathbf{D} &= d + \mathbf{\Gamma} \\ &= dx^\mu(\partial_\mu + \mathbf{\Gamma}_\mu), \end{aligned} \tag{7}$$

where the components of the connection $\mathbf{\Gamma}$ are functions of the metric and its derivatives, known as the Christoffel connection or Christoffel symbol. The transformations properties of $\mathbf{\Gamma}$ under diffeomorphisms are such that the differential operator $\mathbf{D}$ transforms homogeneously (it is covariant) under the diffeomorphism group. To that extent $\mathbf{\Gamma}$ acts as a connection, however this is not enough to turn gravity into a gauge theory. The problem is that this group acts on the coordinates of the manifold as $x^\mu \to x'^\mu(x) = x^\mu + \xi^\mu(x)$, which is a shift in the arguments of the fields (tensors) on which it acts. On the other hand, a gauge transformation in the sense of fiber bundles, acts on the functions and not on their arguments, i.e., it generates a motion along the fiber at a fixed point on the base manifold. For this reason, $\mathbf{\Gamma}$ is not a connection in the sense of fiber bundles.

1.3 Gauge Symmetry and Diffeomorphism Invariance

The difference between gravity and a standard gauge theory for a Yang-Mills system is aggravated by the fact that the action for gravity in four dimensions cannot be written as that of a gauge invariant system for the diffeomorphism group. In YM theories the connection $\mathbf{A}_\mu$ is an element of a Lie algebra $\mathcal{L}$, but the algebraic properties of $\mathcal{L}$ (structure constants and similar invariants, etc.) are independent of the dynamical fields or their equations. In electroweak and strong interactions, the connection A is dynamical, while both the base manifold and the symmetry groups are fixed, regardless of the values of A or the position in spacetime. This implies that the structure constants are neither functions of the field A, or the position x. If $G^a(x)$ are the gauge generators in a YM theory, they obey an algebra of the form

$$[G^a(x), G^b(y)] = C_c^{ab}\delta(x,y)G^c(x), \tag{8}$$

where C_c^{ab} are the structure constants.

In contrast with this, the algebra of diffeomorphisms takes the form

$$\begin{array}{rcl} [\mathcal{H}_\perp(x), \mathcal{H}_\perp(y)] & = & g^{ij}(x)\delta(x,y)_{,i}\mathcal{H}_j(y) - g^{ij}(y)\delta(y,x)_{,i}\mathcal{H}_j(x) \\ [\mathcal{H}_i(x), \mathcal{H}_j(y)] & = & \delta(x,y)_{,i}\mathcal{H}_j(y) - \delta(x,y)_{,j}\mathcal{H}_i(y) \\ [\mathcal{H}_\perp(x), \mathcal{H}_i(y)] & = & \delta(x,y)_{,i}\mathcal{H}_\perp(y) \end{array} \quad , \tag{9}$$

where the $\mathcal{H}_\perp(x)$, $\mathcal{H}_i(x)$ are the generators of time and space translations, respectively, also known as the hamiltonian constraints of gravity, and $\delta(y,x)_{,i} = \frac{\partial\delta(y,x)}{\partial x^i}$ [10]. Clearly, these generators do not form a Lie algebra but an **open algebra**, which has *structure functions* instead of *structure constants* [11]. Here one now finds functions of the dynamical fields, $g^{ij}(x)$ playing the role of the structure constants C_c^{ab}, which identify the symmetry group in a gauge theory. In this case, the structure "constants" may change from one point to another, which means that the symmetry group is not uniformly defined throughout spacetime. This invalidates an interpretation of gravity in terms of fiber bundles, in which the base is spacetime and the symmetry group is the fiber.

1.3.1 Diffeomorphism invariance: a trivial and overrated symmetry

The use of coordinates introduced by Descartes was a great step in mathematics since it allowed to translate geometrical notions into analysis and algebra. The whole Euclidean geometry was reduced to analytic equations. The price was dear, though. Ever since, mathematicians and especially physicists had to live with labels for spacetime points, so much so that space itself was viewed as a number field ($\mathbf{R^3}$). When the geometry of curved surfaces was developed by Gauss, Lobachevsky and Riemann, the use of coordinates became more a nuisance than an advantage as they obscured the intrinsic content of geometric relations.

Tensor calculus was developed as an aid to weed out the meaningful aspects of the geometry from those resulting from the use of particular coordinates: genuine geometrical properties should be insensitive to the changes of coordinate labels. The mere existence of this covariant language goes to show that all reference to coordinates can be consistently hidden from the analysis, in the same way that one can avoid employing units in the derivation of physical laws. Such a langauge exists and it is called exterior calculus, and while it may be necessary at some point to refer to a particular coordinate system –as one may also need to refer to **MKS** units sometimes–, most of the discussion can be made in a coordinate-free manner. An additional advantage in the use of exterior calculus is that since there are no coordinates, there are no coordinate indices in the expressions. This not only makes life easier for TeX fans, but also simplifies formulas and makes their content more apparent.

So, what is the role of diffeomorphism invariance in this language? None. It is built in, in the same way as the invariance under changes of units is built in all physical laws. There is no mathematical content in the former as there is no physical content in the latter. In fact, all of physics and mathematics should be invariant under diffeomorphisms, to the extent that coordinates are human constructs. Lagrange was among the first who realized this and declared that a lagrangian for a mechanical system can be written in any coordinates one may choose and the resulting orbits would not depend on that choice.

2 Gravity as Geometry

Let us now briefly review the standard formulation of General Relativity. On November 25 1915, Albert Einstein presented to the Prussian Academy of Natural Sciences the equations for the gravitational field in the form we now know as Einstein equations [12]. Curiously, five days before, David Hilbert had proposed the correct action principle for gravity, based on a communication in which Einstein had outlined the general idea of what should be the form of the equations [13]. As we shall see, this is not so surprising in retrospect, because there is a unique action which is compatible with the postulates of general relativity in four dimensions that has flat space as a solution. If one allows constant curvature geometries, there is essentially a one-parameter family of actions that can be constructed: the Einstein-Hilbert form plus a cosmological term,

$$I[g] = \int \sqrt{-g}(\alpha_1 R + \alpha_2) d^4x, \tag{10}$$

where R is the scalar curvature, which is a function of the metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$, and its derivatives (for the definitions and conventions used here, see Ref.[14]). The action $I[g]$ is the only functional of the metric which is invariant under general coordinate transformations and gives second order field equations in four dimensions. The coefficients α_1 and α_1 are related to the gravitational constant and the cosmological constant through

$$G = \frac{1}{16\pi\alpha_1}\ ,\ \Lambda = \frac{\alpha_2}{2\alpha_1}. \tag{11}$$

The Einstein equations are obtained by extremizing this action, (10)

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu (R - \Lambda) = 0, \tag{12}$$

and and they are unique in that:

(**i**) they are tensorial equations

(**ii**) they involve only up to second derivatives of the metric

(**iii**) they reproduce Newtonian gravity in the weak field, nonrelativistic approximation.

The first condition implies that the equations have the same meaning in all coordinate systems. This follows from the need to have a coordinate independent (covariant) formulation of gravity in which the gravitational force is replaced by the nonflat geometry of spacetime. The gravitational field being a geometrical entity implies that it cannot depend on the coordinate choice or, in physical terms, a preferred choice of observers.

The second condition means that Cauchy data are necessary (and sufficient in most cases) to integrate the equations. This condition is a concession to the classical physics tradition: the possibility of determining the gravitational field at any moment from the knowledge of the positions and momenta at a given time. This requirement is also the hallmark of Hamiltonian dynamics, which is the starting point for canonical quantum mechanics and therefore suggests that a quantum version of the theory could exist.

The third requirement is the correspondence principle, which accounts for our daily experience that an apple and the moon fall the way they do.

If one further assumes that Minkowski space be among the solutions of the matter-free theory, then one must set $\Lambda = 0$, as most sensible particle physicists would do. If, on the other hand, one believes in static homogeneous and isotropic cosmologies, then Λ must have a finely tuned nonzero value. Experimentally, Λ has a value of the order of 10^{-120} in some "natural" units [15]. Furthermore, astrophysical measurements seem to indicate that Λ must be positive [16]. This presents a problem because there seems to be no theoretical way to predict this "unnaturally small" nonzero value.

As we will see in what follows, for other dimensions the Einstein-Hilbert action is not the only possibility in order to satisfy conditions (**i-iii**).

2.1 Metric and Affine Structures

We conclude this overview by discussing what we mean by spacetime geometry. Geometry is sometimes understood as the set of assertions that can be made about the points on a manifold and their relations. This broad (and vague) idea, is often viewed as encoded in the metric tensor, $g_{\mu\nu}(x)$, which provides the notion of distance between nearby points with coordinates x^μ and $x^\mu + dx^\mu$,

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu. \tag{13}$$

This is the case in Riemannian geometry, where all relevant objects defined on the manifold (distance, area, angles, parallel transport operations, curvature, etc.) can be constructed from the metric. However, it can be distinguished between **metric** and **affine** features of space. *Metricity* refers to lengths, areas, volumes, etc., while *affinity* refers to properties which remain invariant under translations –or more generally, affine transformations–, such as **parallelism**. This distinction is useful because the two notions are logically independent and reducing one to the other is an unnecessary form of violence.

Euclidean geometry was constructed using two elementary instruments: the compass and the (unmarked) straightedge. The first is a metric instrument because it allows comparing lengths and, in particular, drawing circles. The second is used to draw straight lines which, as will be seen below, is a basic affine operation.

Pythagoras' famous theorem is a metric statement; it relates the **lengths** of the sides of a triangle. Affine properties on the other hand, do not change if the scale is changed, as for example the shape of a triangle or, more generally, the angle between two straight lines. A typical affine statement is, for instance, the fact that when two parallel lines intersect a third, the corresponding angles are equal.

Of course parallelism can be reduced to metricity. As we learned in school, one can draw a parallel to a line using a right angled triangle and an unmarked straightedge: One aligns one of the short sides of the triangle with the straight line and rests the other short side on the ruler. Then, one slides the triangle to where the parallel is to be drawn. Thus, given a way to draw right angles and a straight line in space, one can define parallel transport.

As any child knows from the experience of stretching a string or a piece of rubber band, a straight line is the shape of the shortest line between two points. This is clearly a metric feature because it requires *measuring* lengths. Orthogonality is also a metric notion that can be defined using the scalar product obtained from the metric. A right angle is a metric feature because in order to build one, one should be able to *measure* angles, or measure the sides of triangles[3]. We will now argue that, strictly speaking, *parallelism does not require metricity.*

There is something excessive about the construction of a parallel line to another through a given point: one doesn't *have* to use a right angle. In fact, any angle could be used in order to draw a parallel to the first line, so long as it *doesn't change* when we slide it from one point to another. Thus, the essence of parallel transport is a rigid, angle-preserving wedge and a straightedge to connect two points.

There is still some cheating in this argument because we took the construction of a straightedge for granted. What if we had no notion of distance, how do we know what a straight line is? Fortunately, there is a purely affine way

[3]The Egyptians knew how to *use* Pythagoras' theorem to make a right angle, although they didn't know how to prove it. Their recipe was probably known long before, and all good construction workers today still know the recipe: make a loop of rope with 12 segments of equal length. Then, the triangle formed with the loop so that its sides are 3, 4 and 5 segments long is such that the shorter segments are perpendicular to each other [17].

to construct a straight line: Take two short enough segments (two short sticks, matches or pencils would do), and slide them one along the other, as a cross country skier would do. In this way a straight line is generated by *parallel transport of a vector along itself*, and we have not used distance anywhere. It is essentially this *affine* definition of a straight line that can be found in Book I of Euclid's Elements. This definition could be regarded as the *straightest line*, which does not necessarily coincide with the *line of shortest distance*. In fact, if the two sticks one uses are two identical arcs, one could construct families of "straight" lines by parallel transport a segment along itself, but they would not correspond to shortest lines. They are conceptually (i.e., logically) independent objects.

In a space devoid of a metric structure the "straightest" line could be a rather strange looking curve, but it could still be used to define parallelism. Suppose the ruler has been constructed by transporting a vector along itself, then one can use it to define parallel transport as in the standard definition, completely oblivious to the fact that the straight lines are not the shortest, because there would be no way to measure distances. There would be nothing wrong with such construction except that it need not coincide with the standard metric construction.

The fact that this purely affine construction is logically acceptable means that parallel transport is not necessarily a metric concept unless one insists on reducing affinity to metricity.

In differential geometry, parallelism is encoded in the affine connection mentioned earlier, $\Gamma^{\alpha}_{\beta\gamma}(x)$, so that a vector u at the point of coordinates x is said to be parallel to the vector $\tilde{u}$ at a point with coordinates $x+dx$, if their components are related by "parallel transport",

$$\tilde{u}^{\alpha}(x+dx) = \Gamma^{\alpha}_{\beta\gamma} dx^{\beta} u^{\gamma}(x). \tag{14}$$

The affine connection $\Gamma^{\alpha}_{\beta\gamma}(x)$ need not be functionally related to the metric tensor $g_{\mu\nu}(x)$. However, Einstein formulated General Relativity adopting the point of view that the spacetime metric is the only dynamically independent field, while the affine connection is a function of the metric given by the Christoffel symbol,

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda}(\partial_{\beta} g_{\lambda\gamma} + \partial_{\gamma} g_{\lambda\beta} - \partial_{\lambda} g_{\beta\gamma}). \tag{15}$$

This was the starting point of a controversy between Einstein and Cartan, which is vividly recorded in the correspondence they exchanged between May 1929 and May 1932 [18]. In his letters, Cartan insisted politely but forcefully that metricity and parallelism could be considered as independent, while Einstein pragmatically replied that since the space we live in seems to have a metric, it would be more economical to assume the affine connection to be a function of the metric. Cartan advocated economy of assumptions. Einstein argued in favor of economy of independent fields.

Here we adopt Cartan's point of view. It is less economical in dynamical variables but is more economical in assumptions and therefore more general. This alone would not be sufficient reason to adopt Cartan's philosophy, but it turns out to be more transparent in many ways and lends itself better to discuss the differences between gauge theories and gravity.

3 First Order Formulation for Gravity

We view spacetime as a smooth D-dimensional manifold of lorentzian signature M, which at every point x possesses a D-dimensional tangent space T_x. This tangent space T_x is a good approximation of the manifold M in the neighborhood of x. This means that there is a way to represent tensors on M by tensors on T_x. The precise relation between the tensor spaces on M and on T_x is an isomorphism represented by means of a linear mapping, also called *"soldering form"* or simply, *"vielbein"*.

More importantly, the collection of tangent spaces at each point of the manifold makes it possible to define the action of the symmetry group of the tangent space (the Lorentz group) at each point of M, thus endowing it with a fiber bundle structure, the *tangent bundle*. In order to define a derivative operator on the manifold, a connection field is required so that the differential structure remains invariant under local Lorentz transformations that act independently at each spacetime point. This field is the Lorentz connection, also called "spin connection" in the physics literature[4].

3.1 The Vielbein

The translation (isomorphism) between the tensor spaces on M and on T_x is made by means of a dictionary, also called *"soldering form"* or simply, *"vielbein"*. It is sufficient to define its action on a complete set of vectors such as the coordinate separation dx^μ between two infinitesimally close points on M. The corresponding separation dz^a in T_x is defined to be

$$dz^a = e^a_\mu(x)dx^\mu \tag{16}$$

The family $\{e^a_\mu(x), a = 1, ..., D = dimM\}$ can also be seen as a local orthonormal frame on M. The definition (16) makes sense only if the vielbein $e^a_\mu(x)$ transforms as a covariant vector under diffeomorphisms on M and as a contravariant vector under local Lorentz rotations of T_x, $SO(D-1,1)$ (we assumed the signature of the manifold M to be Lorentzian). A similar one to one correspondence can be established between tensors on M and on T_x: if Π is a

[4]Here, only the essential ingredients are given. For a more extended discussion, there are several texts such as those of Refs.[3], [19] and [20] .

tensor with components $\Pi^{\mu_1\cdots\mu_n}$ on M, then the corresponding tensor on the tangent space T_x is[5]

$$P^{a_1\cdots a_n}(x) = e^{a_1}_{\mu_1}(x)\cdots e^{a_n}_{\mu_n}(x)\Pi^{\mu_1\cdots\mu_n}(x). \tag{17}$$

An example of this map between tensors on M and on T_x is the relation between the metrics of both spaces,

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}. \tag{18}$$

This relation can be read as to mean that the vielbein is in this sense the square root of the metric. Given $e^a_\mu(x)$ one can find the metric and therefore, all the metric properties of spacetime are contained in the vielbein. The converse, however, is not true: given the metric, there exist infinitely many choices of vielbein that reproduce the same metric. If the vielbein are transformed as

$$e^a_\mu(x) \longrightarrow e'^a_\mu(x) = \Lambda^a_b(x)e^b_\mu(x), \tag{19}$$

where the matrix $\mathbf{\Lambda}(x)$ leaves the metric in the tangent space unchanged,

$$\Lambda^a_c(x)\Lambda^b_d(x)\eta_{ab} = \eta_{cd}, \tag{20}$$

then the metric $g_{\mu\nu}(x)$ is clearly unchanged. The matrices that satisfy (20) form the Lorentz group $SO(D-1,1)$. This means, in particular, that there are many more components in e^a_μ than in $g_{\mu\nu}$. In fact, the vielbein has D^2 independent components, whereas the metric has only $D(D+1)/2$. The mismatch is exactly $D(D-1)/2$, the number of independent rotations in D dimensions.

3.2 The Lorentz Connection

The Lorentz group acts on tensors at each T_x independently. Thus, the matrices $\mathbf{\Lambda}$ that describe the Lorentz transformations are functions of x. In order to define a derivative of tensors in T_x, one must compensate for the fact that at neighboring points the Lorentz rotations are not the same. This is not different from what happens in any other gauge theory: one needs to introduce a connection for the gauge group. In this case, it is the Lorentz connection, $\omega^a{}_{b\mu}(x)$, defined so that if $\phi^a(x)$ is a field that transforms as a vector under the Lorentz group, $SO(D-1,1)$, its covariant derivative,

$$D_\mu\phi^a(x) = \partial_\mu\phi^a(x) + \omega^a{}_{b\mu}(x)\phi^b(x), \tag{21}$$

[5]The inverse vielbein, $e^\mu_a(x)$, where $e^\nu_a(x)e^b_\nu(x) = \delta^b_a$, and $e^\nu_a(x)e^a_\mu(x) = \delta^\nu_\mu$, relates lower index tensors,

$$P_{a_1\ldots a_n}(x) = e^{\mu_1}_{a_1}(x)\cdots e^{\mu_n}_{a_n}(x)\Pi_{\mu_1\ldots\mu_n}(x).$$

also transforms like a Lorentz vector at x. This requires that under a $SO(D-1,1)$ rotation, $\Lambda^a_c(x)$, the connection transforms as [see (2)]

$$\omega^a_{\ b\mu}(x) \longrightarrow \omega'^a_{\ b\mu}(x) = \Lambda^a_c(x)\Lambda^d_b(x)\omega^c_{\ d\mu}(x) + \Lambda^a_c(x)\partial_\mu\Lambda^c_b(x). \tag{22}$$

In physics, $\omega^a_{\ b\mu}(x)$ is often called the *spin connection*, but Lorentz connection could be a more appropriate name. The word "spin" is due to the fact that $\omega^a_{\ b\mu}$ arises naturally in the discussion of spinors, which carry a special representation of the group of rotations in the tangent space, but that is irrelevant at the moment.

The connection $\omega^a_{\ b\mu}(x)$ defines the *parallel transport* of Lorentz tensors in the tangent space T_x to T_{x+dx}, where x and $x+dx$ are nearby points. The parallel transport of the vector field $\phi^a(x)$ from x to $x+dx$, is a vector $\phi^a_{||}(x+dx)$, defined as

$$\phi^a_{||}(x+dx) \equiv \phi^a(x) + dx^\mu\partial_\mu\phi^a(x) + dx^\mu\omega^a_{\ b\mu}(x)\phi^b(x). \tag{23}$$

Thus, the covariant derivative measures the change in a tensor produced by parallel transport between neighboring points,

$$dx^\mu D_\mu\phi^a(x) = \phi^a_{||}(x+dx) - \phi^a(x). \tag{24}$$

In this way, the affine properties of space are encoded in the components $\omega^a_{\ b\mu}(x)$, which are, until further notice, totally arbitrary and independent from the metric. The group $SO(D-1,1)$ has two invariant tensors, the Minkowski metric, η_{ab}, and the totally antisymmetric Levi-Civitta tensor, $\epsilon_{a_1a_2\cdots s_D}$. Because they are the same in every tangent space, they are constant, and since they are invariant, they are also covariantly constant,

$$d\eta_{ab} = D\eta_{ab} \;=\; 0, \tag{25}$$

$$d\epsilon_{a_1a_2\cdots s_D} = D\epsilon_{a_1a_2\cdots a_D} \;=\; 0. \tag{26}$$

This implies that the Lorentz connection satisfies two identities,

$$\eta_{ac}\omega^c_{\ \ b} \;=\; -\eta_{bc}\omega^c_{\ \ a}, \tag{27}$$

$$\epsilon_{b_1a_2\cdots a_D}\omega^{b_1}_{\ \ a_1} + \epsilon_{a_1b_2\cdots a_D}\omega^{b_2}_{\ \ a_2} + \;\cdots\; + \epsilon_{a_1a_2\cdots b_D}\omega^{b_D}_{\ \ a_D} = 0. \tag{28}$$

The requirement that the Lorentz connection be compatible with the metric structure of the tangent space (27) restricts ω^{ab} to be antisymmetric, while the second relation (28) does not impose further restrictions on the components of the Lorentz connection. Then, the number of independent components of $\omega^a_{\ b\mu}$ is $D^2(D-1)/2$, which is *less* than the number of independent components of the Christoffel symbol $(D^2(D+1)/2)$.

3.3 Differential forms

It can be observed that both the vielbein and the spin connection appear through the combinations

$$e^a(x) \equiv e^a_\mu(x)dx^\mu, \tag{29}$$

$$\omega^a_{\ b}(x) \equiv \omega^a_{\ b\mu}(x)dx^\mu, \tag{30}$$

that is, they are local 1-forms. This is not an accident. It turns out that all the geometric properties of M can be expressed with these two 1-forms and their exterior derivatives only. Since both e^a and $\omega^a_{\ b}$ only carry no coordinate indices (μ, ν, etc.), these 1-forms are scalars under diffeomorphisms on M. Like all exterior forms, they are invariant under coordinate transformations. This is why a description of the geometry that only uses these forms (and their exterior derivatives, defined below) is naturally coordinate-free.

In this formalism the spacetime tensors are replaced by tangent space tensors. In particular, the Riemann curvature 2-form is

$$\begin{aligned} R^a_{\ b} &= d\omega^a_{\ b} + \omega^a_{\ b} \wedge \omega^a_{\ b} \\ &= \frac{1}{2} R^a_{\ b\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \tag{31}$$

where d stands for the 1-form exterior derivative operator, $dx^\mu \partial_\mu \wedge$. The main properties of this operator is that when it acts on a p-form α_p, $d\alpha_p$ is a (p+1)-form, and that,

$$d(d\alpha_p) =: d^2\alpha_p = 0. \tag{32}$$

For a formal definition of this operator and its action see, e.g. [20, 21]. The curvature two form defined by (31) is a Lorentz tensor on the tangent space, and is related to the Riemann tensor, $R^\alpha_{\ \beta\mu\nu}$, through

$$R^a_{\ b} = \frac{1}{2} e^a_{\ \alpha} e^\beta_{\ b} R^\alpha_{\ \beta\mu\nu} dx^\mu \wedge dx^\nu. \tag{33}$$

The fact that $\omega^a_{\ b}(x)$ and the gauge potential in Yang-Mills theory, $A^a_{\ b} = A^a_{\ b\mu} dx^\mu$, are both 1-forms and have similar properties is not an accident since they are both connections of a gauge group[6]. Their transformation laws have the same form and the curvature $R^a_{\ b}$ is completely analogous to the field strength in Yang-Mills,

$$F^a_{\ b} = dA^a_{\ b} + A^a_{\ c} \wedge A^c_{\ b}. \tag{34}$$

There is an asymmetry with respect to the vielbein, though. Its transformation properties under the Lorentz group is not that of a connection but of a vector

[6] In what for physicists is fancy language, ω is a locally defined Lie algebra valued 1-form on M, which is also a connection in the principal $SO(D-1,1)$-bundle over M, while A is a Lie algebra valued 1-form on M, which is also a connection in the vector bundle of some gauge group G.

and the corresponding field in an arbitrary gauge theory would be a matter field. Another important geometric object obtained from derivatives of e^a is the Torsion 2-form,

$$T^a = de^a + \omega^a_{\ b} \wedge e^b, \tag{35}$$

which is a function of the vielbein and the connection and hence depends on both e and ω, while $R^a_{\ b}$ is not a covariant derivative of anything and depends only on ω.

Thus, the basic building blocks of first order gravity are e^a, $\omega^a_{\ b}$, $R^a_{\ b}$, T^a. With them one must put together an action. But, are there other building blocks? The answer is no and the proof is by exhaustion. As a cowboy would put it, if there were any more of them 'round here, we would have heard... And we haven't. However, there is a more subtle argument to rule out the existence of other building blocks. We are interested in objects that transform in a controlled way under Lorentz rotations (vectors, tensors, spinors, etc.). The existence of certain identities implies the taking the covariant derivatives of e^a, $R^a_{\ b}$, and T^a, one finds always combinations of the same objects, or zero:

$$\begin{aligned} De^a &= de^a + \omega^a_{\ b} \wedge e^b = T^a & (36)\\ DR^a_{\ b} &= dR^a_{\ b} + \omega^a_{\ c} \wedge R^c_{\ b} + \omega^c_{\ b} \wedge R^a_{\ c} = 0 & (37)\\ DT^a &= dT^a + \omega^a_{\ b} \wedge T^b = R^a_{\ b} \wedge e^b. & (38) \end{aligned}$$

The first relation is just the definition of torsion and the other two are known as the Bianchi identities. We leave it to the reader to prove these identities, which are direct consequences of the fact that the exterior derivative is nilpotent, $d^2 = \partial_\mu \partial_\nu dx^\mu \wedge dx^\nu = 0$.

In the next sections we discuss the construction of the possible actions for gravity using these ingredients. In particular, in 4 dimensions, the Einstein action can be written as

$$I[g] = \int \epsilon_{abcd}(\alpha R^{ab} e^c e^d + \beta e^a e^b e^c e^d). \tag{39}$$

This is basically the only action for gravity in dimension four, but many more options exist in higher dimensions.

4 Gravity as a Gauge Theory

Symmetry principles help in constructing the right classical action and, more importantly, they are often sufficient to ensure the viability of a quantum theory obtained from the classical action. In particular, local or gauge symmetry is the key to prove consistency (renormalizability) of the field theories we know for the correct description of three of the four basic interactions of nature. The gravitational interaction has stubbornly escaped this rule in spite of the fact that, as we saw, it is described by a theory based on general covariance, which is a local invariance quite analogous to gauge symmetry. In this lecture we try to shed some light on this puzzle.

Roughly one year after C. N. Yang and R. Mills proposed their model for non-abelian gauge invariant interactions [**?**], R. Utiyama showed that the Einstein theory can also be written as a gauge theory for the Lorentz group [22]. This can be checked directly from the lagrangian in (39), which is a Lorentz scalar and hence, trivially invariant under (local) Lorentz transformations. This generated the expectation to construe gravity as a gauge theory for the Poincaré group, $G = ISO(3,1)$, which is the standard symmetry group in particle physics including translations besides the Lorentz transformations. Including the translations seems natural in view of the fact that a general coordinate transformation

$$x^i \to x^i + \xi^i, \tag{40}$$

looks like a local translation. This suggests that diffeomorphism invariance could be identified with the local translations necessary to enlarge the Lorentz group into its Poincaré embedding.

Although this looks plausible, it has been impossible to accommodate the local translations as a gauge symmetry in four dimensions and several attempts to identify the translations with local coordinate transformations have failed. The problem is that no action for general relativity is known which is invariant under any one of these extended groups [23, 24, 25, 26]. In other words, although the fields ω^{ab} and e^a match the generators of the group G, there is no Poincaré-invariant 4-form available constructed with the connection for the Lie algebra of G. We see that although superficially correct, the assertion that gravity is a gauge theory for the translation group is crippled by the profound differences between a gauge theory with fiber bundle structure and another with an open algebra, such as gravity.

A more sophisticated approach could be to replace the Poincaré group by a different G which also contains the Loretnz transformations as a subgroup. The idea is as follows: Our conviction that the space we live in is four dimensional and approximately flat stems from our experience that we can act with the group of four dimensional translations to connect any two points in spacetime. But we know that this could be only approximately true. Like the Earth, our spacetime could be curved but with a radius of curvature so large we wouldn't notice the deviation from flatness except in very delicate observations. So, instead of the symmetries of a four-dimensional *flat* spacetime, we might be experiencing the symmetries of a four dimensional spacetime of *nonzero constant curvature*, also known as a pseudosphere.

Thus it would be natural to expect the local symmetries of spacetime to compatible with a larger group which contains both $SO(3,1)$ and some symmetries analogous to translations,

$$SO(3,1) \hookrightarrow G. \tag{41}$$

The smallest nontrivial choices for G –which are not a direct product of the form $SO(3,1) \times G_0$–, are:

$$G = \begin{cases} SO(4,1) & \text{de Sitter (dS)} \\ SO(3,2) & \text{anti-de Sitter (AdS)} \\ ISO(3,1) & \text{Poincaré} \end{cases} \tag{42}$$

The de Sitter and anti-de Sitter groups are semisimple, while the Poincaré group, which is a contraction of the other two, is not semisimple. This technical detail could mean that $SO(4,1)$ and $SO(3,2)$ have better chances than the Poincaré group, to become physically relevant for gravity. Semisimple groups are preferred as gauge groups because they have an invariant in the group, known as the *Killing metric*, which can be used to define kinetic terms for the gauge fields[7].

In spite of this improved scenario, it is still not possible to write gravity in four dimensions as a gauge theory for the dS or AdS groups. As we shall see next, in odd dimensions ($D = 2n-1$), and only in that case, gravity can be cast as a gauge theory of the groups $SO(D,1)$, $SO(D-1,2)$, or $ISO(D-1,1)$, in contrast with what one finds in dimension four, or in any other even dimension.

5 Lovelock Gravity

We now turn to the construction of an action for gravity which must be a local functional of the one-forms e^a, $\omega^a_{\ b}$ and their exterior derivatives. The fact that $d^2 \equiv 0$ implies that the lagrangian will involve at most first derivatives of these fields through the two-forms $R^a_{\ b}$ and T^a. We need not worry about invariance under general coordinate transformations as exterior forms are coordinate scalars by construction. On the other hand, the action principle cannot depend on the choice of basis in the tangent space and hence Lorentz invariance should be ensured. A sufficient condition to respect Lorentz invariance is to demand the lagrangian to be a Lorentz scalar (although, as we will see, this is not strictly necessary) and for this construction, the two invariant tensors of the Lorentz group, η_{ab}, and $\epsilon_{a_1 \cdots a_D}$ can be used to raise, lower and contract indices.

Finally, since the action must be an integral over the D-dimensional spacetime manifold, the problem is to construct a Lorentz invariant D-form with the following ingredients:

$$e^a, \quad \omega^a_{\ b}, \quad R^a_{\ b}, \quad T^a, \quad \eta_{ab}, \quad \epsilon_{a_1 \cdots a_D}. \tag{43}$$

Thus, we tentatively postulate the lagrangian for gravity to be a D-form made of linear combinations of products of the above ingredients in any possible

[7]Non semisimple groups contain *abelian* invariant subgroups. The generators of the abelian subgroups commute among themselves, and the fact that they are invariant subgroups implies that the too many structure constants in the Lie algebra vanish, which in turn makes the Killing metric to acquire zero eigenvalues preventing its invertibility.

way so as to form a Lorentz scalar. We exclude from the ingredients functions such as the metric and its inverse, which rules out the Hodge $\star$-dual. The only justification for this is that: **i**) it reproduces the known cases, and **ii**) it explicitly excludes inverse fields, like $e^{\mu}_{a}(x)$, which would be like A^{-1}_{μ} in Yang-Mills theory (see [27] and [28] for more on this). This postulate rules out the possibility of including tensors like the Ricci tensor $R_{\mu\nu} = \eta_{ac}e^{c}_{\mu}e^{\lambda}_{b}R^{ab}_{\ \ \lambda\nu}$, or $R_{\alpha\beta}R_{\mu\nu}R^{\alpha\mu\beta\nu}$, etc. That this is sufficient and necessary to account for all sensible theories of gravity in D dimensions is the contents of a theorem due to David Lovelock [29], which in modern language can be stated thus:

Theorem [Lovelock, 1970 [29]-Zumino, 1986 [27]]: The most general action for gravity that does not involve torsion and gives at most second order field equations for the metric [8], is of the form

$$I_D = \kappa \int \sum_{p=0}^{[D/2]} \alpha_p L^{(D,p)}, \tag{44}$$

where the α_ps are arbitrary constants, and $L^{(D,p)}$ is given by

$$L^{(D,\,p)} = \epsilon_{a_1\cdots a_d} R^{a_1a_2} \cdots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \cdots e^{a_D}. \tag{45}$$

Here and in what follows we omit the wedge symbol in the exterior products.

For $D = 2$ this action reduces to a linear combination of the 2-dimensional Euler character, χ_2, and the spacetime volume (area),

$$\begin{aligned} I_2 &= \kappa \int \alpha_1 \epsilon_{ab} R^{ab} + \alpha_0 \epsilon_{ab} e^a e^b \\ &= \kappa \int \sqrt{|g|}\,(\alpha_1 R + 2\alpha_0)\, d^2x \\ &= \kappa\alpha_1 \cdot \chi_2 + 2\kappa\alpha_0 \cdot V_2. \end{aligned} \tag{46}$$

This action has only one local extremum, $V = 0$, which reflects the fact that, unless other matter sources are included, I_2 does not make a very interesting dynamical theory for the geometry. If the geometry has a prescribed boundary the first term picks up a boundary term and the action is extremized by a surface that takes the shape of a soap bubble. This is the famous Plateau problem, which consists of establishing the shape of the surface of minimal area bounded by a certain fixed closed curve (see, e., g., http://www-gap.dcs.st-and.ac.uk/history/Mathematicians/Plateau.html).

For $D = 3$, (44) reduces to the Hilbert action with a volume term, whose coefficient is the cosmological constant. For $D = 4$ the action has, in addition,

[8] These conditions can be translated to mean that the Lovelock theories possess the same degrees of freedom as the Einstein Hilbert lagrangian in each dimension, that is, $D(D-3)/2$ (see, e.g., [30])

the four dimensional Euler invariant χ_4,

$$\begin{aligned} I_4 &= \kappa \int \alpha_2 \epsilon_{abcd} R^{ab} R^{cd} + \alpha_1 \epsilon_{abcd} R^{ab} e^c e^d + \alpha_0 L^{(4,\,0)} \\ &= \kappa \int \sqrt{|g|} \left[\alpha_2 \left(R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4 R^{\alpha\beta} R_{\alpha\beta} + R^2 \right) + 2\alpha_1 R + 24\alpha_0 \right] d^4x \\ &= \kappa\alpha_2 \cdot \chi_4 + 2\alpha_1 \int \sqrt{|g|} R d^4x + 24\kappa\alpha_0 \cdot V_4. \end{aligned} \tag{47}$$

For all dimensions the lagrangian is a polynomial of degree $d \leq D/2$ in the curvature 2-form. In general, each term $L^{(D,\,p)}$ in the lagrangian (44) is the continuation to D dimensions of the Euler density from dimension $p \leq D$ [27]. In particular, for even D the highest power in the curvature is the Euler character χ_D. In four dimensions, the term $L^{(4,\,2)}$ in (47) can be identified as the Gauss-Bonnet density, whose integral over a closed compact four dimensional manifold M_4 equals the Euler characteristic $\chi(M_4)$. This term also provides the first nontrivial generalization of Einstein gravity occurring in five dimensions, where the quadratic term that can be added to the lagrangian is the 5-form

$$\epsilon_{abcde} R^{ab} R^{cd} e^e = \sqrt{|g|} \left[R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4 R^{\alpha\beta} R_{\alpha\beta} + R^2 \right] d^5x. \tag{48}$$

It seems that it had been known for many years that Gauss-Bonnet term could be added to the Einstein-Hilbert action in five dimensions [31]. The generalization to arbitrary D in the form (44) was obtained more than 30 years ago in [29] and is known as the Lovelock lagrangian,

$$L_D = \sum_{p=0}^{[D/2]} \alpha_p L^{(D,p)}. \tag{49}$$

This lagrangian was also identified as describing the only ghost-free [9] effective theory for a spin two field, generated from string theory at low energy [32, 27]. From our perspective, the absence of ghosts is only a reflection of the fact that the Lovelock action yields at most second order field equations for the metric, so that the propagators behave as αk^{-2}, and not as $\alpha k^{-2} + \beta k^{-4}$, as would be the case in a general higher derivative theory.

5.1 Dynamical Content

Extremizing the action (44) with respect to e^a and ω^{ab}, yields

$$\delta I_D = \int [\delta e^a \mathcal{E}_a + \delta\omega^{ab} \mathcal{E}_{ab}] = 0, \tag{50}$$

[9] Physical states in quantum field theory have positive probability, which means that they are described by positive norm vectors in a Hilbert space. Ghosts instead, are unphysical states of negative norm. A lagrangian containing arbitrarily high derivatives of fields generally leads to ghosts. The fact that a gravitational lagrangian such as (49) leads to a ghost-free theory was unexpected and is highly nontrivial.

modulo surface terms. The condition for I_D to have an extreme under arbitrary first order variations is that the terms $\mathcal{E}_a$ $\mathcal{E}_{ab}$ vanish. This implies that the geometry should satisfy the equations

$$\mathcal{E}_a = \sum_{p=0}^{[\frac{D-1}{2}]} \alpha_p (d-2p)\mathcal{E}_a^{(p)} = 0, \tag{51}$$

and

$$\mathcal{E}_{ab} = \sum_{p=1}^{[\frac{D-1}{2}]} \alpha_p p(d-2p)\mathcal{E}_{ab}^{(p)} = 0, \tag{52}$$

where we have defined

$$\mathcal{E}_a^{(p)} := \quad \epsilon_{ab_2\cdots b_{D-1}} R^{b_2 b_3} \cdots R^{b_{2p} b_{2p+1}} e^{b_{2p+1}} \cdots e^{b_D}, \tag{53}$$

$$\mathcal{E}_{ab}^{(p)} := \quad \epsilon_{aba_3\cdots a_d} R^{a_3 a_4} \cdots R^{a_{2p-1} a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \cdots e^{a_D}. \tag{54}$$

These equations involve only first derivatives of e^a and $\omega^a_{\ b}$, simply because $d^2 = 0$. If one furthermore assumes –as is usually done– that the torsion vanishes identically,

$$T^a = de^a + \omega^a_{\ b} e^b = 0, \tag{55}$$

then Eq. (54) is automatically satisfied and (55) can be solved for ω as a function of the inverse vielbein (e^μ_a) and its derivative as

$$\omega^a_{\ b\mu} = -e^\nu_b(\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a)_\lambda), \tag{56}$$

where $\Gamma^\lambda_{\mu\nu}$ is symmetric in $\mu\nu$ and can be identified as the Christoffel symbol (torsion-free affine connection). Substituting this expression for the spin connection back into (53) yields second order field equations for the metric. These equations are identical to the ones obtained from varying the Lovelock action written in terms of the Riemann tensor and the metric,

$$I_D[g] = \int d^D x \sqrt{g} \left[\alpha'_o + \alpha'_1 R + \alpha'_2 (R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2) + \cdots \right]. \tag{57}$$

This purely metric form of the action is also called second order formalism, because it contains up to second derivatives of the metric. The fact that the lagrangian contains second derivatives of $g_{\mu\nu}$ has induced some authors to refer to these actions as *higher derivative theories of gravity.* This, however, is incorrect. The Einstein-Hilbert action, as well as its Lovelock generalization, both yield second order field equations and for the same reason: the second derivatives of the metric enter through a total derivative in the lagrangian and therefore the field equations remain second order. Lagrangians that do give rise to higher order field equations for the metric are those that contain arbitrary powers of the curvature tensor and their contractions, like R^s, with $s \neq 1$, or $\left(\alpha R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + bR^{\alpha\beta}R_{\alpha\beta} + cR^2\right)$, with $a : b : c \neq 1 : -4a : 1$, etc.

Higher derivatives equations for the metric would mean that the initial conditions required to uniquely determine the time evolution are not those of General Relativity and hence the theory would have different degrees of freedom from standard gravity. It would also make the propagators in the quantum theory to develop poles at imaginary energies: *ghosts*. Ghost states spoil the unitarity of the theory, making it hard to make sense of it and to interpret its predictions.

One important feature of the Lovelock theories, that makes their behavior very different for $D \leq 4$ and for $D > 4$ is that in the former case the field equations (51,52) are linear in R^{ab}, while in the latter case the equations are nonlinear in the curvature tensor. In particular, while for $D \leq 4$ the equations (54) imply the vanishing of torsion, this is no longer true for $D > 4$. In fact, the field equations evaluated in some configurations may leave some components of the curvature and torsion tensors completely undetermined. For example, Eq.(52) has the form of a polynomial in R^{ab} times T^a, and it is possible that the polynomial vanishes identically, imposing no conditions on the torsion tensor. However, the configurations for which the equations do not determine R^{ab} and T^a form sets of measure zero in the space of geometries. In a generic case, outside of these degenerate configurations, the Lovelock theory has the same $D(D-3)/2$ degrees of freedom as in ordinary gravity [33].

5.2 Adding Torsion

Lovelock's theorem assumes torsion to be identically zero. If equation (55) is assumed as an identity, it means that e^a and $\omega^a_{\ b}$ are not independent fields, contradicting the assumption that these fields correspond to two independent features of the geometry on equal footing. Moreover, for $D \leq 4$, equation (55) coincides with (54), so that imposing the torsion-free constraint is at best unnecessary.

In general, if the field equation for some field ϕ can be solved algebraically as $\phi = f(\psi)$ in terms of the remaining fields, then by the implicit function theorem, the original action principle $I[\phi, \psi]$ is identical to the reduced one obtained by substituting $f(\psi)$ in the action, $I[f(\psi), \psi]$. This occurs in 3 and 4 dimensions, where the spin connection can be algebraically obtained from its own field equation and $I[\omega, e] = I[\omega(e, \partial e), e]$. In higher dimensions, however, the torsion-free condition is not necessarily a consequence of the field equations and although (54) is algebraic in ω, it is practically impossible to solve for ω as a function of e. Therefore, it is not clear in general whether the action $I[\omega, e]$ is equivalent to the second order form of the action, $I[\omega(e, \partial e), e]$.

It turns out that the torsion-free condition does not automatically follow from the field equation (52). It could be that the curvature is such that the torsion is completely indeterminate, as it happens for instance if the geometry has constant curvature for the choice of α's to be discussed below. Therefore, it seems natural to consider the generalization of the Lovelock action in which torsion *is not* assumed to vanish identically. This generalization consists of adding of all possible Lorentz invariants involving T^a explicitly. This includes combinations like $R^{ab}e_b$, which do not involve torsion explicitly but which vanish for $T = 0$ ($DT^a = R^{ab}e_b$). The general construction was worked out in [34]. The main difference with the torsion-free case is that now, apart from the dimensional continuation of the Euler densities, one encounters the Pontryagin (or Chern) classes as well.

For $D = 3$, the only new torsion term not included in the Lovelock family is

$$e^a T_a, \tag{58}$$

while for $D = 4$, there are three terms not included in the Lovelock series,

$$e^a e^b R_{ab}, \;\; T^a T_a, \;\; R^{ab} R_{ab}. \tag{59}$$

The last term in (59) is the Pontryagin density, whose integral also yields a topological invariant. A linear combination of the other two terms is a topological invariant known as the Nieh-Yan density, given by [35]

$$N_4 = T^a T_a - e^a e^b R_{ab}. \tag{60}$$

The properly normalized integral of (60) over a 4-manifold is an integer related to the $SO(5)$ and $SO(4)$ Pontryagin classes [36].

In general, the terms related to torsion that can be added to the action are combinations of the form

$$\begin{aligned}
A_{2n} &= e_{a_1} R^{a_1}_{\;\;a_2} R^{a_2}_{\;\;a_3} \cdots R^{a_{n-1}}_{\;\;a_n} e^{a_n}, \text{even n} \geq 2 && (61)\\
B_{2n+1} &= T_{a_1} R^{a_1}_{\;\;a_2} R^{a_2}_{\;\;a_3} \cdots R^{a_{n-1}}_{\;\;a_n} e^{a_n}, \text{any n} \geq 1 && (62)\\
C_{2n+2} &= T_{a_1} R^{a_1}_{\;\;a_2} R^{a_2}_{\;\;a_3} \cdots R^{a_{n-1}}_{\;\;a_n} T^{a_n}, \text{odd n} \geq 1 && (63)
\end{aligned}$$

which are $2n$, $2n + 1$ and $2n + 2$ forms, respectively. These Lorentz invariants belong to the same family with the Pontryagin densities or Chern classes,

$$P_{2n} = R^{a_1}_{\;\;a_2} R^{a_2}_{\;\;a_3} \cdots R^{a_n}_{\;\;a_1}, \text{even n} \geq 2. \tag{64}$$

The lagrangians that can be constructed now are much more varied and there is no uniform expression that can be provided for all dimensions. For example, in 8 dimensions, in addition to the Lovelock terms, one has all possible 8-forms made by taking products among the elements of the set $\{A_4, \, A_8, \, B_3, \, B_5, \, B_7, \, C_4, \, C_8, \, P_4, \, P_8\}$. They are

$$(A_4)^2, A_8, (B_3 B_5), (A_4 C_4), (C_4)^2, C_8, (A_4 P_4), (C_4 P_4), (P_4)^2, P_8. \tag{65}$$

To make life even harder, there are some linear combinations of these products which are topological densities, as in (59). In 8 dimensions there are two

Pontryagin forms

$$
\begin{aligned}
P_8 &= R^{a_1}_{\ a_2} R^{a_2}_{\ a_3} \cdots R^{a_4}_{\ a_1}, \\
(P_4)^2 &= (R^a_{\ b} R^b_{\ a})^2,
\end{aligned}
$$

which occur also in the absence of torsion, and there are also two generalizations of the Nieh-Yan form,

$$
\begin{aligned}
(N_4)^2 &= (T^a T_a - e^a e^b R_{ab})^2, \\
N_4 P_4 &= (T^a T_a - e^a e^b R_{ab})(R^c_{\ d} R^d_{\ c}),
\end{aligned}
$$

etc. (for details and extensive discussions, see Ref.[34]).

6 Selecting Sensible Theories

Looking at these expressions one can easily feel depressed. The lagrangians look awkward and the number of terms in them grows wildly with the dimension[10]. This problem is not purely aesthetic. The coefficients in front of each term in the lagrangian are arbitrary and dimensionful. This problem already occurs in 4 dimensions, where the cosmological constant has dimensions of $[\text{length}]^{-4}$ and, as evidenced by the outstanding cosmological constant problem, there is no theoretical argument to fix its value in order to compare with the observations.

The presence of dimensionful parameters leaves little room for optimism in a quantum version of the theory. Dimensionful parameters in the action are potentially dangerous because they are likely to give rise to uncontrolled quantum corrections. This is what makes ordinary gravity nonrenormalizable in perturbation theory: In 4 dimensions, Newton's constant has dimensions of length (or inverse mass) squared in natural units. This means that as the order in perturbation theory increases, more powers of momentum will occur in the Feynman graphs, making the ultraviolet divergences increasingly worse. Concurrently, the radiative corrections to these bare parameters require the introduction of infinitely many counterterms into the action to render them finite[6]. But an illness that requires infinite amount of medication is synonym of incurable.

The only safeguard against the threat of uncontrolled divergences in the quantum theory is to have some symmetry principle that fixes the values of the parameters in the action, limiting the number of possible counterterms that could be added to the lagrangian. Thus, if the symmetry that fixes the values of the parameters will protect these values. A good indication that this might happen would be if all the coupling constants are dimensionless and could be absorbed in the fields, as in Yang-Mills theory.

[10]As it is shown in [34], the number of torsion-dependent terms grows as the partitions of $D/4$, which is given by the Hardy-Ramanujan formula, $p(D/4) \sim \frac{1}{\sqrt{3D}} \exp[\pi\sqrt{D/6}]$.

As shown below, in odd dimensions there is a unique combination of terms in the action that can give the theory an enlarged gauge symmetry. The resulting action can be seen to depend on a unique multiplicative coefficient (κ), analogous to Newton's constant. Moreover, this coefficient can be shown to be quantized by a argument similar to Dirac's quantization of the product of magnetic and electric charge [37].

6.1 Extending the Lorentz Group

The coefficients α_p in the Lovelock lagrangian (49) have dimensions l^{D-2p}. This is because the canonical dimension of the vielbein is $[e^a] = l^1$, while the Lorentz connection has dimensions that correspond to a true gauge field, $[\omega^{ab}] = l^0$. This reflects the fact that gravity is naturally only a gauge theory for the Lorentz group, where e^a plays the role of a matter field, while the vielbein *is not* a connection field but transforms as a vector under Lorentz rotations.

6.1.1 Poincaré Group

Three-dimensional gravity is an important exception to this statement where e^a does play the role of a connection. Consider the Einstein-Hilbert lagrangian in three dimensions,

$$L_3 = \epsilon_{abc} R^{ab} e^c. \tag{66}$$

Under an infinitesimal Lorentz transformation with parameter $\lambda^a{}_b$, the Lorentz connection transforms as

$$\begin{aligned} \delta\omega^a{}_b &= D\lambda^a{}_b \\ &= d\lambda^a{}_b + \omega^a{}_c \lambda^c{}_b - \omega^c{}_b \lambda^a{}_c, \end{aligned} \tag{67}$$

while e^c, R^{ab} and ϵ_{abc} transform as tensors,

$$\begin{aligned} \delta e^a &= \lambda^a{}_c e^c \\ \delta R^{ab} &= \lambda^a{}_c R^{cb} + \lambda^b{}_c R^{ac}, \\ \delta \epsilon_{abc} &= \lambda^d{}_a \epsilon_{dbc} + \lambda^d{}_b \epsilon_{adc} + \lambda^d{}_c \epsilon_{abd} \equiv 0. \end{aligned}$$

Combining these relations, the Lorentz invariance of L_3 can be shown directly. What is unexpected is that e^a can be viewed as a gauge connection for the translation group. In fact, the vielbein can be assumed to transforms under "local translations" in tangent space, parametrized d by λ^a, as a connection:

$$\begin{aligned} \delta e^a &= D\lambda^a \\ &= d\lambda^a + \omega^a{}_b \lambda^b. \end{aligned} \tag{68}$$

Then, the lagrangian L_3 changes by a total derivative,

$$\delta L_3 = d[\epsilon_{abc} R^{ab} \lambda^c], \tag{69}$$

which can be dropped under standard boundary conditions. This means that, in three dimensions ordinary gravity can be considered as a gauge theory of the Poincaré group. We leave it as an exercise to the reader to prove this[11]

6.1.2 (Anti-)de Sitter Group

It is also possible to do this in the presence of a cosmological constant $\Lambda = \mp\frac{1}{6l^2}$. Now the lagrangian (49) reads

$$L_3^{AdS} = \epsilon_{abc}(R^{ab}e^c \pm \frac{1}{3l^2}e^a e^b e^c), \quad (70)$$

and the action is invariant –modulo surface terms– under the infinitesimal transformations,

$$\begin{aligned} \delta\omega^{ab} &= [d\lambda^{ab} + \omega^a{}_c\lambda^{cb} + \omega^b{}_c\lambda^{ac}] \mp [e^a\lambda^b - \lambda^a e^b]l^{-2} && (71) \\ \delta e^a &= [\lambda^a{}_b e^b] \; + \; [d\lambda^a + \omega^a{}_b\lambda^b]. && (72) \end{aligned}$$

These transformations can be cast in a more suggestive way as

$$\begin{aligned} \delta\begin{bmatrix} \omega^{ab} & e^a l^{-l} \\ -e^b l^{-l} & 0 \end{bmatrix} &= d\begin{bmatrix} \lambda^{ab} & \lambda^a l^{-l} \\ -\lambda^b l^{-l} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \omega^a_{\ c} & \pm e^a l^{-l} \\ -e_c l^{-l} & 0 \end{bmatrix}\begin{bmatrix} \lambda^{cb} & \lambda^c l^{-1} \\ -\lambda^b l^{-1} & 0 \end{bmatrix} \\ &+ \begin{bmatrix} \omega^b_{\ c} & \pm e^b l^{-1} \\ -e_c l^{-1} & 0 \end{bmatrix}\begin{bmatrix} \lambda^{ac} & \lambda^a l^{-1} \\ -\lambda^c l^{-1} & 0 \end{bmatrix}. \end{aligned}$$

This can also be written as

$$\delta W^{AB} = dW^{AB} + W^A_C\Lambda^{CB} + W^B_C\Lambda^{AC},$$

where the 1-form W^{AB} and the 0-form Λ^{AB} stand for the combinations

$$\begin{aligned} W^{AB} &= \begin{bmatrix} \omega^{ab} & e^a l^{-1} \\ -e^b l^{-1} & 0 \end{bmatrix} && (73) \\ \Lambda^{AB} &= \begin{bmatrix} \lambda^{ab} & \lambda^a l^{-1} \\ -\lambda^b l^{-1} & 0 \end{bmatrix}, && (74) \end{aligned}$$

(here $a, b, .. = 1, 2, ..D$, while $A, B, ... = 1, 2, .., D+1$). Clearly, W^{AB} transforms as a connection and Λ^{AB} can be identified as the infinitesimal transformation parameters, but for which group? A clue comes from the fact that $\Lambda^{AB} = -\Lambda^{BA}$. This immediately indicates that the group is one that leaves invariant a symmetric, real bilinear form, so it must be a group in the $SO(r, s)$ family.

[11]Hint: use the infinitesimal transformations δe and $\delta\omega$ to compute the commutators of the second variations to obtain the Lie algebra of the Poincaré group.

The signs ($\pm$) in the transformation above can be traced back to the sign of the cosmological constant. It is easy to check that this structure fits well if indices are raised and lowered with the metric

$$\Pi^{AB} = \left[\begin{array}{cc} \eta^{ab} & 0 \\ 0 & \pm 1 \end{array} \right], \tag{75}$$

so that, for example, $W^{AB} = \Pi^{BC} W^{A}\ C$. Then, the covariant derivative in the connection W of this metric vanishes identically,

$$D_W \Pi^{AB} = d\Pi^{AB} + W^{A}_{C} \Pi^{CB} + W^{B}_{C} \Pi^{AC} = 0. \tag{76}$$

Since Π^{AB} is constant, this last expression implies $W^{AB} + W^{BA} = 0$, in exact analogy with what happens with the Lorentz connection, $\omega^{ab} + \omega^{ba} = 0$, where $\omega^{ab} \equiv \eta^{bc} \omega^{a}_{\ c}$. Indeed, this is a very awkward way to discover that the 1-form W^{AB} is actually a connection for the group which leaves invariant the metric Π^{AB}. Here the two signs in Π^{AB} correspond to the de Sitter (+) and anti-de Sitter ($-$) groups, respectively.

What we have found here is an explicit way to immerse the three-dimensional Lorentz group into a larger symmetry group, in which the vielbein has been promoted to a component of a larger connection, on equal footing with the Lorentz connection.

The Poincaré symmetry is obtained in the limit $l \to \infty$. In that case, instead of (71, 72) one has

$$\delta\omega^{ab} = [d\lambda^{ab} + \omega^{a}_{\ c} \lambda^{cb} + \omega^{b}_{\ c} \lambda^{ac}] \tag{77}$$

$$\delta e^{a} = [\lambda^{a}_{\ b} e^{b}] \ + \ [d\lambda^{a} + \omega^{a}_{\ b} \lambda^{b}]. \tag{78}$$

In this limit, which is the familiar symmetry group of Minkowski space, the representation in terms of W becomes inadequate because the metric Π^{AB} becomes degenerate (noninvertible) and is not clear how to raise and lower indices anymore. Nevertheless, the lagrangian (70) in the limit$l \to \infty$ takes the usual Einstein Hilbert form with vanishing cosmological constant,

$$L_3^{EH} = \epsilon_{abc} R^{ab} e^{c}, \tag{79}$$

which can be directly checked to be invariant under (78). We leave this as an exercise for the reader.

As Witten showed, GR in three spacetime dimensions is a renormalizable quantum system [4]. It is strongly suggestive that precisely in 2+1 dimensions this is also a gauge theory on a fiber bundle. It could be thought that the exact solvability miracle is due to the absence of propagating degrees of freedom in three-dimensional gravity, but the final power-counting argument of renormalizability rests on the fiber bundle structure of the Chern-Simons system and doesn't seem to depend on the absence of propagating degrees of freedom. In what follows we will generalize the gauge invariance of 3 dimensional gravity to higher dimensions.

6.2 More Dimensions

Everything that has been said about embedding the Lorentz group into the (A)dS group for $D = 3$, starting at equation (71), can be generalized for any D. In fact, it is always possible to embed the D-dimensional Lorentz group into the de-Sitter, or anti-de Sitter groups,

$$SO(D-1,1) \hookrightarrow \begin{cases} SO(D,1), & \Pi^{AB} = \mathrm{diag}(\eta^{\mathrm{ab}}, +1) \\ SO(D-1,2), & \Pi^{AB} = \mathrm{diag}(\eta^{\mathrm{ab}}, -1) \end{cases} . \tag{80}$$

as well as into their Poincaré limit,

$$SO(D-1,1) \hookrightarrow ISO(D-1,1). \tag{81}$$

The question naturally arises: can one find an action for gravity in other dimensions which is also invariant, not just under the Lorentz group, but under one of its extensions, $SO(D,1)$, $SO(D-1,2)$, $ISO(D-1,1)$? As we will see now, the answer to this question is affirmative in odd dimensions. There is always a action for $D = 2n-1$, invariant under local $SO(2n-2,2)$, $SO(2n-1,1)$ or $ISO(2n-2,1)$ transformations, in which the vielbein and the spin connection combine to form the connection of the larger group. In even dimensions, however, this cannot be done.

Why is it possible in three dimensions to enlarge the symmetry from local $SO(2,1)$ to local $SO(3,1)$, $SO(2,2)$ and $ISO(2,1)$? What happens if one tries to do this in four or more dimensions?

Let us start with the Poincaré group and the Hilbert action for $D = 4$,

$$L_4 = \epsilon_{abcd} R^{ab} e^c e^d. \tag{82}$$

Why is this not invariant under local translations $\delta e^a = d\lambda^a + \omega^a{}_b \lambda^b$? A simple calculation yields

$$\begin{aligned} \delta L_4 &= 2\epsilon_{abcd} R^{ab} e^c \delta e^d \\ &= d(2\epsilon_{abcd} R^{ab} e^c \lambda^d) + 2\epsilon_{abcd} R^{ab} T^c \lambda^d. \end{aligned} \tag{83}$$

The first term in the r.h.s. of (83) is a total derivative and therefore gives a surface contribution to the action. The last term, however, need not vanish, unless one imposes the field equation $T^a = 0$. But this means that the invariance of the action only occurs on shell. Now, "on shell symmetries" are not real symmetries and they probably don't survive quantization because quantum mechanics doesn't respect equations of motion.

On the other hand, the miracle occurred in 3 dimensions because the lagrangian (79) is linear in e. In fact, a lagrangian of the form

$$L_{2n+1} = \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \tag{84}$$

is also invariant under local Poincaré transformations (77, 78), as can be easily checked. Since the Poincaré group is a limit of (A)dS, it seems likely that there should exist a lagrangian in odd dimensions, invariant under local (A)dS transformations, whose limit for vanishing cosmological constant ($l \to \infty$) is (84). One way to find out what that lagrangian might be, one could take the most general Lovelock lagrangian and select the coefficients by requiring invariance under (71, 72). This is a long and tedious but sure route. An alternative approach is to try to understand why it is that in three dimensions the gravitational lagrangian with cosmological constant (70) is invariant under the (A)dS group.

Let us consider the three-dimensional case first. If we take seriously the notion that W^{AB} is a connection, then the associated curvature is

$$F^{AB} = dW^{AB} + W^A_C W^{CB},$$

where W^{AB} is defined in (73). Then, it is easy to prove that

$$F^{AB} = \left[\begin{array}{cc} R^{ab} \pm l^{-2} e^a e^b & l^{-1} T^a \\ -l^{-1} T^b & 0 \end{array} \right]. \tag{85}$$

where a, b run from 1 to 3 and A, B from 1 to 4. Since the (A)dS group has an invariant tensor ϵ_{ABCD}, one can construct the 4-form invariant

$$E_4 = \epsilon_{ABCD} F^{AB} F^{CD}. \tag{86}$$

This is invariant under the (A)dS group and is readily recognized as the Euler density for a four-dimensional manifold[12] whose tangent space is not Minkowski, but has the metric Π^{AB} =diag $(\eta^{ab}, \mp 1)$. The Euler density E_4 can also be written explicitly in terms of R^{ab}, T^a, and e^a,

$$\begin{aligned} E_4 &= 4\epsilon_{abc}(R^{ab} \pm l^{-2} e^a e^b) l^{-1} T^a \qquad (87) \\ &= \frac{4}{l} d \left[\epsilon_{abc} \left(R^{ab} \pm \frac{1}{3l^2} e^a e^b \right) e^a \right], \end{aligned}$$

which is, up to constant factors, the exterior derivative of the three-dimensional lagrangian (70),

$$E_4 = \frac{4}{l} dL_3^{AdS}. \tag{88}$$

This is the key point: the l.h.s. of (88) is invariant (by construction) under local (A)dS; so, the same must be true of the r.h.s.,

$$\delta \left(dL_3^{AdS} \right) = 0.$$

[12]This identification is formal, since the differential forms that appear here are defined in three dimensions, but they can be naturally extended to four dimensional forms by simply extending the range of coordinate indices. This implies that one is considering the three-dimensional manifold as embedded in –or, better, as the boundary of– a four dimensional manifold.

Since the variation (δ) is a linear operation, this can also be written as

$$d\left(\delta L_3^{AdS}\right) = 0,$$

which in turn means, by Poincaré's Lemma [38] that, locally, $\delta L_3^{AdS} = d(something)$. This explains why the action is (A)dS invariant up to surface terms, which is exactly what we found for the variation, [see, (69)]. The fact that three dimensional gravity can be written in this way was observed many years ago in Refs. [39, 4].

The key to generalize the (A)dS lagrangian from 3 to $2n-1$ dimensions is now clear[13]:

• First, generalize the Euler density (86) to a $2n$-form,

$$E_{2n} = \epsilon_{A_1 \cdots A_{2n}} F^{A_1 A_2} \cdots F^{A_{2n-1} A_{2n}}. \tag{89}$$

• Second, express E_{2n} explicitly in terms of R^{ab}, T^a and e^a using (85).

• Write this as the exterior derivative of a $(2n-1)$-form L_{2n-1}.

• L_{2n-1} can be used as a lagrangian in $(2n-1)$ dimensions and is (A)dS invariant by construction.

Proceeding in this way, directly yields the $(2n-1)$-dimensional (A)dS invariant lagrangian as

$$L_{2n-1}^{(A)dS} = \sum_{p=0}^{n-1} \bar{\alpha}_p L^{(2n-1,p)}, \tag{90}$$

where $L^{(D,p)}$ is given by (45). This is a particular case of a Lovelock lagrangian in which all the coefficients $\bar{\alpha}_p$ have been fixed to take the values

$$\bar{\alpha}_p = \kappa \cdot \frac{(\pm 1)^{p+1} l^{2p-D}}{(D-2p)} \begin{pmatrix} n-1 \\ p \end{pmatrix}, \; p = 1, 2, ..., n-1 = \frac{D-1}{2}, \tag{91}$$

where $1 \leq p \leq n-1 = (D-1)/2$, and κ is an arbitrary dimensionless constant. It is left as an exercise to the reader to check that $dL_{2n+1}^{(A)dS} = E_{2n}$ and to show the invariance of $L_{2n-1}^{(A)dS}$ under the (A)dS group.

Another exercise would be to show that the action (90) can also be written as [44]

$$I_{2n+1}^{Reg} = \kappa \int_M \int_0^1 dt \, \epsilon R_t^n e^{a_{2n+1}} \;, \tag{92}$$

where we have suppressed al Lorentz indices, and $R_t^{ab} := R^{ab} + t^2 e^a e^b$.

Example: In five dimensions, the (A)dS lagrangian reads

$$L_5^{(A)dS} = \kappa \cdot \epsilon_{abcde} \left[\frac{1}{l} e^a R^{bc} R^{de} \pm \frac{2}{3l^3} e^a e^b e^c R^{de} + \frac{1}{5l^5} e^a e^b e^c e^d e^e \right]. \tag{93}$$

[13] The construction we outline here was discussed by Chamseddine [40], Müller-Hoissen [41], and by Bañados, Teitelboim and this author in [42, 43].

The parameter l is a length scale –the Planck length– and cannot be fixed by theoretical considerations. Actually, l only appears in the combination

$$\tilde{e}^a = \frac{e^a}{l},$$

that could be considered as the "true" dynamical field, as is the natural thing to do if one uses W^{AB} instead of ω^{ab} and e^a separately. In fact, the lagrangian (90) can also be written in terms of W^{AB} and its exterior derivative, as

$$L^{(A)dS}_{2n-1} = \kappa \cdot \epsilon_{A_1 \cdots A_{2n}} \left[W(dW)^{n-1} + a_3 W^3 (dW)^{n-2} + \cdots a_{2n-1} W^{2n-1} \right], \quad (94)$$

where all indices are contracted appropriately and the coefficients a_3, a_5, are all dimensionless combinatoric factors.

6.3 Chern-Simons

There is a more general way to look at these lagrangians in odd dimensions, which also sheds some light on their remarkable enlarged symmetry. This is summarized in the following

Lemma: Let $\mathcal{C}(\mathcal{F})$ be an invariant $2n$-form constructed with the field strength $F = dA + A^2$, where A is the connection for some gauge group G. If there exists a $2n-1$ form, L, depending on A and dA, such that $dL = \mathcal{C}$, then under a gauge transformation, L changes by a total derivative (exact form), $\delta L = d$(something).

The $(2n-1)-$form L is known as the Chern-Simons (**CS**) lagrangian. This lemma shows that L defines a nontrivial lagrangian for A which*is not invariant* under gauge transformations, but that changes by a function that only depends on the fields at the boundary. This is sufficient to define a physical lagrangian since the action principle considers variations of the physical fields subject to some set of appropriate boundary conditions. So, it is always possible to select the condition on the fields at the boundary in such that $\delta L = 0$.

This construction is not restricted to the Euler invariant discussed above, but applies to any invariant of similar nature, generally known as characteristic classes. Other well known characteristic classes are the Pontryagin or Chern classes and their corresponding CS forms were studied first in the context of abelian and nonabelian gauge theories (see, e. g., [45, 3]).

The following table gives examples of CS forms which define lagrangians in three dimensions, and their corresponding topological invariants,

$D = 3$ Chern-Simons Lagrangian	Top. Invariant($\mathcal{C}$)	Group
$L^{(A)dS}_3 = \epsilon_{abc}(R^{ab} \pm \frac{e^a e^b}{3l^2})e^c$	$\mathcal{E}_4 = \epsilon_{abc}(R^{ab} \pm \frac{e^a e^b}{l^2})T^c$	$SO(4)\dagger$
$L^{Lorentz}_3 = \omega^a_{\ b} d\omega^b_{\ a} + \frac{2}{3}\omega^a_{\ b}\omega^b_{\ c}\omega^c_{\ a}$	$\mathcal{P}^{Lorentz}_4 = R^a_{\ b} R^b_{\ a}$	$SO(2,1)$
$L^{Torsion}_3 = e^a T_a$	$\mathcal{N}_4 = T^a T_a - e^a e^b R_{ab}$	$SO(2,1)$
$L^{U(1)}_3 = AdA$	$\mathcal{P}^{U(1)}_4 = FF$	$U(1)$
$L^{SU(N)}_3 = tr[\mathbf{A}d\mathbf{A}+\frac{2}{3}\mathbf{AAA}]$	$\mathcal{P}^{\ SU(4)}_{\ 4} = tr[\mathbf{FF}]$	$SU(N)$

†Either this or any of its cousins, $SO(3,1)$, $SO(2,2)$

In this table, R, F, and $\mathbf{F}$ are the curvatures of the Lorentz connection $\omega^a_{\ b}$, the electromagnetic connection A, and the Yang-Mills ($SU(N)$) connection $\mathbf{A}$, respectively; T is the torsion, $\mathcal{E}_4$ is the Euler density, $\mathcal{P}_4$ is the Pontryagin density for the Lorentz group [21], and $\mathcal{N}_4$ is the Nieh-Yan invariant [35]. The lagrangians are locally invariant (up to total derivatives) under the corresponding gauge groups.

6.4 Torsional Chern-Simons

So far we have not included torsion in the CS lagrangians, but as we see in the table above, it is also possible to construct CS forms that include torsion. All the CS forms above are Lorentz invariant (up to an exact form), but there is a linear combination of the second and third which is invariant under the (A)dS group. This is the so-called exotic gravity [4],

$$L_3^{Exotic} = L_3^{Lor} + \frac{2}{l^2} L_3^{Tor}. \tag{95}$$

As can be shown directly by taking its exterior derivative, this is invariant under (A)dS:

$$\begin{aligned} dL_3^{Exotic} &= R^a_{\ b} R^b_{\ a} \pm \frac{2}{l^2}\left(T^a T_a - e^a e^b R_{ab}\right) \\ &= F^A_{\ B} F^B_{\ A}. \end{aligned}$$

This exotic lagrangian has the curious property of giving exactly the same field equations as the standard dL_3^{AdS}, but interchanged: varying with respect to e^a one gives the equation for ω^{ab} of the other.

In five dimensions, there are no Lorentz invariants that can be formed using T^a and hence no new torsional lagrangians. In seven dimensions there are three torsional CS terms,

$D = 7$ Torsional Chern-Simons Lagrangian	$\mathcal{C}$
$L_7^{Lorentz} = \omega(d\omega)^3 + \cdots + \frac{4}{7}\omega^7$	$R^a_{\ b}R^b_{\ c}R^c_{\ d}R^d_{\ a}$
$L_7^A = (L_3^{Lorentz})R^a_{\ b}R^b_{\ a} = \left(\omega^a_{\ b}d\omega^b_{\ a} + \frac{2}{3}\omega^a_{\ b}\omega^b_{\ c}\omega^c_{\ a}\right) R^a_{\ b}R^b_{\ a}$	$\left(R^a_{\ b}R^b_{\ a}\right)^2$
$L_7^B = (L_3^{Torsion})R^a_{\ b}R^b_{\ a} = e^aT_aR^a_{\ b}R^b_{\ a}$	$(T^aT_a - e^ae^bR_{ab})R^d_{\ c}R^c_{\ d}$

6.5 Characteristic Classes and Even D

The CS construction fails in $2n$ dimensions for the simple reason that there are no topological invariants constructed with the ingredients we are using in $2n+1$ dimensions. The topological invariants we have found so far, also called characteristic classes, are the Euler and the Pontryagin or Chern-Weil classes. The idea of characteristic class is one of the unifying concepts in mathematics and it connects algebraic topology, differential geometry and algebraic geometry. The theory of characteristic classes explains mathematically why it is not always possible to perform a gauge transformation that makes the connection vanish everywhere even if it is locally pure gauge. The non vanishing value of a topological invariant signals an obstruction to the existence of a gauge transformation that trivializes the connection globally.

There are basically two types of invariants relevant for a Lorentz invariant theory in an even-dimensional manifold

• The Euler class, associated with the $O(D-n, n)$ groups. In two dimensions, the Euler number is related to the genus (g) of the surface, $\chi = 2 - 2g$.

• The Pontryagin class, associated with any classical semisimple group G. It counts the difference between self dual and anti-self dual gauge connections that are admitted in a given manifold.

The Nieh-Yan invariants correspond to the difference between Pontriagin classes for $SO(D+1)$ and $SO(D)$ in D dimensions [36].

As there are no similar invariants in odd dimensions, there are no CS actions for gravity for even D, invariant under the (anti-) de Sitter or Poincaré groups. In this light, it is fairly obvious that although ordinary Einstein-Hilbert gravity can be given a fiber bundle structure for the Lorentz group, this structure cannot be extended to include local translational invariance.

6.5.1 Quantization of the gravitation constant

The only free parameter in the action is κ. Suppose this lagrangian is used to describe a simply connected, compact $2n-1$ dimensional manifold M, which is the boundary of a $2n$-dimensional compact orientable manifold Ω. Then the action for the geometry of M can be expressed as the integral of the Euler density E_{2n} over Ω, multiplied by κ. But since there can be many different manifolds with the same boundary M, the integral over Ω should give the physical predictions as that over another manifold, Ω'. In order for this change to leave the path integral unchanged, a minimal requirement would be

$$\kappa \left[\int_{\Omega} E_{2n} - \int_{\Omega'} E_{2n} \right] = 2n\pi\hbar. \tag{96}$$

The quantity in brackets –with right normalization– is the Euler number of the manifold obtained by gluing Ω and Ω' along M in the right way to produce an orientable manifold, $\chi[\Omega \cup \Omega']$. This integral can take an arbitrary integer value and from this one concludes that κ must be quantized[37],

$$\kappa = nh, \tag{97}$$

where h is Planck's constant.

6.5.2 Born-Infeld gravity

The closest one can get to a CS theory in even dimensions is with the so-called Born-Infeld (**BI**) theories [42, 43, 46]. The BI lagrangian is obtained by a particular choice of the α_p's in the Lovelock series, so that the lagrangian takes the form

$$L_{2n}^{BI} = \epsilon_{a_1 \cdots a_{2n}} \bar{R}^{a_1 a_2} \cdots \bar{R}^{a_{2n-1} a_{2n}}, \tag{98}$$

where $\bar{R}^{ab}$ stands for the combination

$$\bar{R}^{ab} = R^{ab} \pm \frac{1}{l^2} e^a e^b. \tag{99}$$

With this definition it is clear that the lagrangian (98) contains only one free parameter, l, which, as we have explained, can always be absorbed in a redefinition of the vielbein. This lagrangian has a number of interesting classical features like simple equations, black hole solutions, cosmological models, etc. [42, 43, 47]. The simplification comes about because the equations admit a unique maximally symmetric configuration given by $\bar{R}^{ab} = 0$, in contrast with the situation when all α_p's are arbitrary. As we have mentioned, for arbitrary α_p's, the field equations do not determine completely the components of R^{ab} and T^a in general. This is because the high nonlinearity of the equations can give rise to degeneracies. The BI choice is in this respect the best behaved since the degeneracies are restricted to only one value of the radius of curvature ($R^{ab} \pm \frac{1}{l^2} e^a e^b = 0$). At the same time, the BI action has the least number of algebraic constrains required by consistency among the field equations, and it is therefore the one with the simplest dynamical behavior[46].

6.6 Finite Action and the Beauty of Gauge Invariance

Classical lagrangians are defined modulo total derivatives because surface terms are usually assumed to vanish under variations. However, this is true for the simplest type of boundary conditions, namely those that keep the values of the fields fixed at the boundary: Dirichlet conditions. In a gauge theory, however, it may be more relevant to fix gauge invariant properties at the boundary –like the curvature–, which are not precisely boundary condition of the Dirichlet type.

On the other hand, it is also desirable to have an action which is finite when evaluated at a physically observable configuration. This is not just for elegance, it is a necessity if one wants to study the semiclassical thermodynamic description of the theory. This is particularly true for a theory that possesses black holes, with interesting thermodynamic properties.

In [44] it is shown that the action has an extremum when the field equations hold, and is finite on classically interesting configuration if the AdS action (90) is supplemented with a boundary term of the form

$$B_{2n} = -\kappa n \int_0^1 dt \int_0^t ds\, \epsilon\theta e \left(\widetilde{R} + t^2\theta^2 + s^2 e^2\right)^{n-1}, \tag{100}$$

where $\widetilde{R}$ and θ are the intrinsic and extrinsic curvatures of the boundary. The resulting action attains an extremum for boundary conditions that fix the extrinsic curvature of the boundary. In that reference is also shown that this action principle yields finite charges (mass) without ad-hoc regularizations or background subtractions.

6.7 Further Extensions

By embedding the Lorentz group into one of its parents, the de Sitter or anti-de Sitter group, or its uncle, the Poincaré group, we have generated a gauge theory for the spacetime geometry for any odd dimension. This theory is based on the affine (ω) and metric (e) features of the spacetime manifold as the only dynamical fields of the system. The theory has no dimensionful couplings and is the natural continuation of gravity in 2+1 dimensions. From here one can go on to study these theories, analyzing its classical solutions, study the cosmologies and black holes that live on them, etc. Although for the author of these notes, black hole solutions in these and related theories have been a constant source of surprises [48, 43, 47, 49], we will not take this path here since that would sidetrack us into a completely different industry.

One could legitimately go on to investigate embeddings in other, larger groups. The Results in this direction are rather disappointing. One could embed the Lorentz group $SO(D-1,1)$ in any $SO(n,m)$, if $n \geq D-1$ and $m \geq 1$, and contractions of them analogous to the limit of vanishing cosmological constant limit that yields the Poincaré group. There are also some accidents like the (local) identity between $SO(3)$ and $SU(2)$, which occur occasionally and which make some people smile and others explode with hysterical joy, but that is rare.

The only other natural generalization of the Lorentz group into a larger group, seems to be direct products with other groups. That yields theories constructed as mere sums of Chern-Simons actions and not very interesting otherwise. The reason for this boring scenario, as we shall see in the next chapters, is connected to the so-called *No-Go Theorems* [50]. Luckily, there is a very remarkable (and at the time revolutionary) way out of this murky situation that is provided by supersymmetry. In fact, despite all the propaganda and false expectations that this unobserved symmetry has generated, this is its most remarkable feature, and possible its only lasting effect in our culture: it provides a natural way to unify the symmetries of spacetime, with internal symmetries like the gauge invariance of electrodynamics, the weak and the strong interactions.

PART TWO

7 Chern-Simons Supergravity

So far we have dealt with the possible ways in which pure gravity can be extended by relaxing three standard assumptions of General Relativity:
i) that the notion of parallelism is derived from metricity,
ii) four-dimensional spacetime, and
iii) lagrangian given by the Einstein Hilbert term $\sqrt{-g}R$ only.
Instead, we demanded:
iv) second order field equations for the metric components,
v) that the lagrangian be a D-form constructed out of the vielbein, e^a, the spin connection, ω^a_b, and their exterior derivatives, and
vi) invariance of the action under local Lorentz rotations in the tangent space.

In this way a family of lagrangians containing higher powers of the curvature and torsion multiplied by arbitrary and dimensionful coefficients are admissible. The embarrassing presence of these arbitrary constants was cured by enlarging the symmetry group, thereby making the theory gauge invariant under the larger symmetry group and simultaneously fixing all parameters in the lagrangian. The cure works only in odd dimensions. The result was a highly nonlinear Chern-Simons theory of gravity, invariant under local (A)dS transformations in the tangent space. We now turn to the problem of enlarging the contents of the theory to allow for supersymmetry. This will have two effects: it will incorporate fermions and fermionic generators into the picture, and it will enlarge the symmetry by including additional bosonic generators. These additional bosonic symmetries are required by consistency (the algebra must close) and are the most important consequence of supersymmetry.

7.1 Supersymmetry

Supersymmetry (**SUSY**) is a curious symmetry: most theoreticians are willing to accept it as a legitimate feature of nature, although it has never been experimentally observed. The reason is that it is such a unique and beautiful idea that it is commonly felt that it would be a pity if it is not somehow realized in nature. This symmetry mixes **bosons** (integer spin particles) and **fermions** (half integer spin particles). These two types of particles obey very different statistics and play very different roles in nature, so it is somewhat surprising that there should exist a symmetry connecting them[14].

[14]Bosons obey Bose-Einstein statistics and, like ordinary classical particles, there is no limit to the number of them that can occupy the same state. Fermions, instead, cannot occupy the same quantum state more than one at a time, something known as Fermi-Dirac statistics. All elementary particles are either bosons or fermions and they play different roles: fermions like electrons, protons, neutrons and quarks are the constituents of matter, while the interactions

The simplest supersymmetric theories combine bosons and fermions on equal footing, rotating into each other under SUSY transformations. This is possibly the most intriguing –and uncomfortable– aspect of supersymmetry: the blatant fact that bosons and fermions play such radically different roles in nature means that SUSY is not manifest around us, and therefore, it must be badly broken at the scale of our observations. Unbroken SUSY would predict the existence of a fermionic carriers of interaction and bosonic constituents of matter as partners of the known particles. None of these two types of particles have been observed and there is no clue at present as to how to break supersymmetry.

In spite of this "lack of realism", SUSY gained the attention of the high energy community mainly because it offered the possibility of taming the ultraviolet divergences of many field theories. It was observed early on that the UV divergences of the bosons were often cancelled out by divergences coming from the fermionic sector. This possibility seemed particularly attractive in the case of a quantum theory of gravity, and in fact, it was shown that in a supersymmetric extension of general relativity, dubbed supergravity (**SUGRA**) the ultraviolet divergences at the one-loop level exactly cancelled (see [51] and references therein). This is one of the most remarkable features of SUSY: local (gauge) SUSY is not only compatible with gravity. In fact, by consistency, local SUSY *requires* gravity.

A most interesting aspect of SUSY is its ability to combine *bosonic spacetime symmetries*, like Poincaré invariance, with other *internal bosonic* symmetries like the $SU(3) \times SU(2) \times U(1)$ invariance of the standard model. Thus SUSY supports the hope that it could be possible to understand the logical connection between spacetime and internal invariances. SUSY makes the idea that these different bosonic symmetries might be related, and in some way necessitate each other, more natural. In this way it might be possible to understand why it is that some internal symmetries are observed and others not. The most important lesson from supersymmetry is not the unification of bosons and fermions, but the extension of the bosonic symmetry.

From an algebraic point of view, SUSY is the simplest nontrivial way to enlarge the Poincaré group, unifying spacetime and internal symmetries, thus circumventing an important obstruction found by S. Coleman and J. Mandula [50]. The obstruction also called *no-go theorem*, roughly states that any Lie algebra that contains the Poincaré and some internal symmetry algebra must be a direct sum of the two [52, 53]. SUSY is nontrivial because the algebra *is not* a direct sum of the spacetime and internal symmetries. The way SUSY circumvents this obstacle is by having both commutators (antisymmetric product, $[\cdot,\cdot]$) and anticommutators (symmetric product, $\{\cdot,\cdot\}$), forming what is known as a **graded Lie algebra**, also called super Lie algebra or simply, superalgebra. For a general introduction to SUSY, see [54, 53].

are described by gauge fields made out of bosons like the photon, the gluon or the $W^{\pm}$ and Z^0, or the graviton.

7.2 Superalgebras

A superalgebra has two types of generators: bosonic, $\mathbf{B}_i$, and fermionic, $\mathbf{F}_\alpha$. They are closed under the (anti-)commutator operation, which follows the general pattern

$$\begin{aligned} \left[\mathbf{B}_i, \mathbf{B}_j\right] &= C_{ij}^k \mathbf{B}_k \quad (101)\\ \left[\mathbf{B}_i, \mathbf{F}_\alpha\right] &= C_{i\alpha}^\beta \mathbf{F}_\beta \quad (102)\\ \{\mathbf{F}_\alpha, \mathbf{F}_\beta\} &= C_{\alpha\beta}^\gamma \mathbf{B}_\gamma \quad (103)\end{aligned}$$

The generators of the Poincaré group are included in the bosonic sector, and the $\mathbf{F}_\alpha$'s are the supersymmetry generators. This algebra, however, does not close for an arbitrary bosonic group. In other words, given a Lie group with a set of bosonic generators, it is not always possible to find a set of fermionic generators to enlarge the algebra into a closed superalgebra. The operators satisfying relations of the form (101-103), are still required to satisfy a consistency condition, the super-Jacobi identity, which is required by associativity,

$$[\mathbf{G}_\mu, [\mathbf{G}_\nu, \mathbf{G}_\lambda]_\pm]_\pm + (-)^{\sigma(\nu\lambda\mu)}[\mathbf{G}_\nu, [\mathbf{G}_\lambda, \mathbf{G}_\mu]_\pm]_\pm + (-)^{\sigma(\lambda\mu\nu)}[\mathbf{G}_\lambda, [\mathbf{G}_\mu, \mathbf{G}_\nu]_\pm]_\pm = 0. \quad (104)$$

Here $\mathbf{G}_\mu$ represents any generator in the algebra, $[R, S]_\pm = RS \pm SR$, where this sign is chosen according the bosonic or fermionic nature of the operators in the bracket, and $\sigma(\nu\lambda\mu)$ is the number of permutations of fermionic generators.

As we said, starting with a set of bosonic operators it is not always possible to find a set of $\mathcal{N}$ fermionic ones that generate a closed superalgebra. It is often the case that extra bosonic generators are needed to close the algebra, and this usually works for some values of $\mathcal{N}$ only. In other cases there is simply no supersymmetric extension at all [55]. This happens, for example, with the de Sitter group, which has no supersymmetric extension in general [53]. For this reason in what follows we will restrict to AdS theories. The general problem of classifying all possible superalgebras that extend the classical Lie algebras has been discussed in [56].

7.3 Supergravity

The name supergravity (**SUGRA**) applies to any of a number of supersymmetric theories that include gravity in their bosonic sectors[15]. The invention/discovery of supergravity in the mid 70's came about with the spectacular announcement that some ultraviolet divergent graphs in pure gravity were cancelled by the inclusion of their supersymmetric partners [51]. For some time it

[15]There are some purists who would reserve this name for supersymmetric theories whose gravitational sector is the Einstein-Hilbert lagrangian. This narrow point seems untenable for dimensions $D > 4$ in view of the variety of possibilities. Our point of view here is that there can be more than one system that can be called supergravity, although its connection with the standard theory remains unsettled.

was hoped that the nonrenormalizability of GR could be cured in this way by its supersymmetric extension. However, the initial hopes raised by SUGRA as a way of taming the ultraviolet divergences of pure gravity eventually vanished with the realization that SUGRAs would be nonrenormalizable as well [57].

Again, one can see that the standard form of SUGRA is not a gauge theory for a group or a supergroup, and that the local (super-)symmetry algebra closes naturally on shell only. The algebra could be made to close off shell by force, at the cost of introducing auxiliary fields –which are not guaranteed to exist for all d and $\mathcal{N}$ [58]–, and still the theory would not have a fiber bundle structure since the base manifold is identified with part of the fiber. Whether it is the lack of fiber bundle structure the ultimate reason for the nonrenormalizability of gravity remains to be proven. It is certainly true, however, that if GR could be formulated as a gauge theory, the chances for its renormalizability would clearly increase. At any rate, now most high energy physicists view supergravity as an effective theory obtained from string theory in some limit. In string theory, eleven dimensional supergravity is seen as an effective theory obtained from ten dimensional string theory at strong coupling [59]. In this sense supergravity would not be a fundamental theory and therefore there is no reason to expect that it should be renormalizable.

7.4 From Rigid Supersymmetry to Supergravity

Rigid or global SUSY is a supersymmetry in which the group parameters are constants throughout spacetime. In particle physics the spacetime is usually assumed to have fixed Minkowski geometry. Then the relevant SUSY is the supersymmetric extension of the Poincaré algebra in which the supercharges are "square roots" of the generators of spacetime translations, $\{\bar{\mathbf{Q}}, \mathbf{Q}\} \sim \Gamma \cdot \mathbf{P}$. The extension of this to a local symmetry can be done by substituting the momentum $\mathbf{P}_\mu = i\partial_\mu$ by the generators of spacetime diffeomorphisms, $\mathcal{H}_\mu$, and relating them to the supercharges by $\{\bar{\mathbf{Q}}, \mathbf{Q}\} \sim \Gamma \cdot \mathcal{H}$. The resulting theory has a local supersymmetry algebra which only closes on-shell [51]. As we discussed above, the problem with on-shell symmetries is that they are not likely to survive quantization.

An alternative approach for constructing the SUSY extension of the AdS symmetry is to work on the tangent space rather than on the spacetime manifold. This point of view is natural if one recalls that spinors are defined relative to a local frame on the tangent space and not as tensors on the coordinate basis. In fact, spinors provide an irreducible representation for $SO(N)$, but not for $GL(N)$, which describe infinitesimal general coordinate transformations. The basic strategy is to reproduce the 2+1 "miracle" in higher dimensions. This idea was applied in five dimensions [40], as well as in higher dimensions [60, 61, 62].

7.5 Standard Supergravity

In its simplest version, supergravity was conceived almost thirty years ago, as a quantum field theory whose action included the Einstein-Hilbert term representing a massless spin-2 particle (graviton), plus a Rarita-Scwinger kinetic term describing a massless spin-3/2 massless particle (gravitino) [63]. These fields would transform into each other under local supersymmetry. Later on, the model was refined, including more "realistic" features, like matter couplings, enlarged symmetries, higher dimensions with their corresponding reductions to four dimensions, cosmological constant, etc., [51]. In spite of the number of variations on the theme, a few features remained as the hallmark of SUGRA, which were a reflection of this history. In time, these properties have become a sort of identikit of SUGRA, although they should not be taken as a set of necessary postulates. Among these we can indicate three that will be relaxed in our construction:

(i) The fermionic and bosonic fields in the lagrangian come in combinations such that they have equal number of propagating degrees of freedom.

(ii) Gravitons are described by the Hilbert action (plus a possible cosmological constant), and,

(iii) The spin connection and the vielbein are not independent fields but are related through the torsion equation.

The first feature is inherited from rigid supersymmetric theories in Minkowski space, where particles form vector representations of the Poincaré group labelled by their spin and mass, and the fields often form vector representations of the internal groups (multiplets). This is justified in a Minkowski background because particle states are represented by the in- and out- plane waves of a weakly interacting theory. This argument, however, breaks down if the Poincaré group in not a symmetry of the theory, as it happens in an asymptotically AdS spacetimes and in other simple cases such as in 1+1 dimensions, with broken translational invariance [64].

The argument goes as follows: The generator of translations in Minkowski space, $P_\mu = (E, \mathbf{p})$, commutes with all symmetry generators, therefore an internal symmetry should only mix particles of equal mass. Since supersymmetry changes the spin by 1/2, a supersymmetric multiplet must contain, for each bosonic eigenstate of the hamiltonian $|E>_B$, a fermionic one with the same energy, $|E>_F = \mathbf{Q}\ |E>_B$, and vice versa. Thus, it seems natural that in supergravity this would still be the case. In AdS space, however, the momentum operator is not an abelian generator, but acts like the rest of Lorentz generators and therefore the supersymmetry generator $\mathbf{Q}$ need not commute with it. Another limitation of this assumption is the fact that it does not consider the possibility that the fields belong to a different representation of the Poincaré or AdS group, such as the adjoint representation.

Also implicit in the argument for counting the degrees of freedom is the usual assumption that the kinetic terms are those of a minimally coupled gauge theory, a condition that is not met by a CS theory. Apart from the difference in background, which requires a careful treatment of the unitary irreducible representations of the asymptotic symmetries [65], the counting of degrees of freedom in CS theories follows a completely different pattern [66] from the counting for the same connection 1-forms in a YM theory [30].

The other two issues concern the purely gravitational sector and are dictated by economy: after Lovelock's theorem, there is no reason to adopt **(ii)**, and the fact that the vielbein and the spin connection are dynamically independent fields on equal footing makes assumption **(iii)** unnatural. Furthermore, the elimination of the spin connection from the action introduces the inverse vielbein in the action and thereby entangling the action of the spacial symmetries defined on the tangent space. The fact that the supergravity generators do not form a closed off-shell algebra may be traced back to these assumptions.

8 AdS Superalgebras

In order to construct a supergravity theory that contains gravity with a cosmological constant, a mathematically oriented physicist would look for the smallest superalgebra that contains the generators of the AdS algebra. This question was asked –and answered– many years ago, at least for some dimensions $D = 2, 3, 4$ *mod* 8, in [55]. But this is not all, we would also like to have an action that realizes the symmetry.

Several supergravities are known for all dimensions $D \leq 11$ [67]. For $D = 4$, a supergravity action that includes a cosmological constant was first discussed in [68], however, finding a supergravity with cosmological constant in an arbitrary dimension is a nontrivial task. For example, the standard supergravity in eleven dimensions without cosmological constant has been know for a long time [69], however, it does not contain a cosmological constant term, and it has been shown to be impossible to accommodate one [70][71]. Moreover, although it was known to the authors of Ref.[69] that the supergroup that contains the AdS group in eleven dimensions is $SO(32|1)$, no action was found for almost twenty years for the theory of gravity which exhibits this symmetry.

In what follows, we present an explicit construction of the superalgebras that contain AdS algebra $so(D-1, 2)$ along the lines of [55] where we have extended the method to apply it to the cases $D = 5$, 7 and 9 as well [61]. The crucial observation is that the Dirac matrices provide a natural representation of the AdS algebra in any dimension. Thus, the AdS connection $\mathbf{W}$ can be written in this representation as $\mathbf{W} = e^a \mathbf{J}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab}$, where

$$\mathbf{J}_a = \frac{1}{2}(\Gamma_a)^\alpha_\beta, \tag{105}$$

$$\mathbf{J}_{ab} = \frac{1}{2}(\Gamma_{ab})^\alpha_\beta. \tag{106}$$

Here $\mathbf{\Gamma}_a$, $a = 1, ..., D$ are $m \times m$ Dirac matrices, where $m = 2^{[D/2]}$ (here $[r]$ denotes the integer part of r), and $\mathbf{\Gamma}_{ab} = \frac{1}{2}[\mathbf{\Gamma}_a, \mathbf{\Gamma}_b]$. These two classes of matrices form a closed commutator subalgebra (the AdS algebra) of the **Dirac algebra** $\mathcal{D}$, obtained by taking only the antisymmetrized products of $\mathbf{\Gamma}$ matrices

$$\mathbf{I}, \mathbf{\Gamma}_a, \mathbf{\Gamma}_{a_1 a_2}, ..., \mathbf{\Gamma}_{a_1 a_2 \cdots a_D}, \tag{107}$$

where

$$\mathbf{\Gamma}_{a_1 a_2 \cdots a_k} = \frac{1}{k!}(\mathbf{\Gamma}_{a_1} \mathbf{\Gamma}_{a_2} \cdots \mathbf{\Gamma}_{a_k} \pm [permutations]).$$

For even D, the matrices in the set (107) are all linearly independent, but for odd D they are not, because $\mathbf{\Gamma}_{12 \cdots D} = \sigma \mathbf{I}$ and therefore half of them are proportional to the other half. Thus, the dimension of this algebra, that is, the number of independent matrices of the form (107) is $m^2 = 2^{2[D/2]}$. This representation provides an elegant way to generate all $m \times m$ matrices (note however, that $m = 2^{[D/2]}$ is not any number).

8.1 The Fermionic Generators

The supersymmetric extension of a given Lie algebra is a mathematical problem whose solution lies in the general classification of superalgebras [56]. Although their representations were studied more than 20 years ago in [55], the application to construct SUSY field theory actions has not been pursued much. Instead of approaching this problem as a general question of classification of irreducible representations, we will take a more practical course, by identifying the representation we are interested in from the start. This representation is the one in which the bosonic generators take a form like (105, 106). The simplest extension of the matrices (105, 106) is obtained by the addition of one row and one column, as

$$\mathbf{J}_a = \begin{bmatrix} \frac{1}{2}(\Gamma_a)^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix}, \tag{108}$$

$$\mathbf{J}_{ab} = \begin{bmatrix} \frac{1}{4}(\Gamma_{ab})^\alpha_\beta & 0 \\ 0 & 0 \end{bmatrix}. \tag{109}$$

The generators associated to the new entries would have only one spinor index. Let us call $\mathbf{Q}_\gamma$ ($\gamma = 1, ..., m$) the generator that has only one nonvanishing entry in the γ-th row of the last column,

$$\mathbf{Q}_\gamma = \begin{bmatrix} 0 & \delta^\alpha_\gamma \\ -C_{\gamma\beta} & 0 \end{bmatrix} \qquad \alpha, \beta = 1, ..., m. \tag{110}$$

Since this generator carries a spinorial index (γ), it is in a spin-1/2 representation of the Lorentz group.

The entries of the bottom row ($C_{\gamma\beta}$) will be so chosen as to produce the smallest supersymmetric extensions of AdS. There are essentially two ways restrict the dimension of the representation compatible with Lorentz invariance: *chirality* and *reality*. In odd dimensions there is no chirality because corresponding "γ_5" is proportional to the identity matrix. Reality instead can be defined in any dimension and refers to whether a spinor and its conjugate are proportional up to a constant matrix, $\bar{\psi} = \mathbf{C}\psi$, or more explicitly,

$$\bar{\psi}^{\alpha} = C^{\alpha\beta}\psi_{\beta}. \tag{111}$$

A spinor that satisfies this condition is said to be Majorana, and $\mathbf{C} = (C^{\alpha\beta})$ is called the charge conjugation matrix. This matrix is assumed to be invertible, $C_{\alpha\beta}C^{\beta\gamma} = \delta^{\gamma}_{\alpha}$, and plays the role of a metric in the space of Majorana spinors.

Using the form (110) for the supersymmetry generator, its Majorana conjugate $\bar{\mathbf{Q}}$ is found to be

$$\begin{aligned} \bar{\mathbf{Q}}^{\gamma} &= C^{\alpha\beta}\mathbf{Q}_{\beta} \\ &= \begin{bmatrix} 0 & C^{\alpha\gamma} \\ -\delta^{\gamma}_{\beta} & 0 \end{bmatrix}. \end{aligned} \tag{112}$$

The matrix $\mathbf{C}$ can be viewed as performing a change of basis $\psi \rightarrow \psi^{\mathrm{T}} = \mathbf{C}\psi$, which is turn corresponds to the change $\Gamma \rightarrow \Gamma^{\mathrm{T}}$. Now, since the Clifford algebra for the Dirac matrices,

$$\{\mathbf{\Gamma}^a, \mathbf{\Gamma}^b\} = 2\eta^{ab}, \tag{113}$$

is also obeyed by their transpose, $(\mathbf{\Gamma}^a)^{\mathrm{T}}$, these two algebras must be related by a change of basis up to a sign,

$$(\mathbf{\Gamma}^a)^{\mathrm{T}} = \eta\mathbf{C}\mathbf{\Gamma}^a\mathbf{C}^{-1} \;\text{ with } \eta^2 = 1. \tag{114}$$

The basis of this Clifford algebra (113) for which an operator $\mathbf{C}$ satisfying (114) exists, is called the Majorana representation. This last equation is the defining relation for the charge conjugation matrix, and whenever it exists, it can be chosen to have definite parity[16],

$$\mathbf{C}^{\mathrm{T}} = \lambda\mathbf{C}, \text{with} \lambda = \pm 1. \tag{115}$$

8.2 Closing the Algebra

We already encountered the bosonic generators responsible for the AdS transformations (105, 106), which have the general form required by (101). It is straightforward to check that commutators of the form $[\mathbf{J}, \mathbf{Q}]$ turn out to be

[16] The Majorana reality condition can be satisfied in any D provided the spacetime signature is such that, if there are s spacelike and t timelike dimensions, then $s - t = 0, 1, 2, 6, 7$ mod 8 [54, 53]. Thus, for lorentzian signature, Majorana spinors can be defined unambiguously only for $D = 2, 3, 4, 8, 9$, mod 8.

proportional to $\mathbf{Q}$, in agreement with the general form (102). What is by no means trivial is the closure of the anticommutator $\{\mathbf{Q},\mathbf{Q}\}$ as in (103). Direct computation yields

$$\begin{aligned} \{\mathbf{Q}_\gamma,\mathbf{Q}_\lambda\}^\alpha_\beta &= \begin{bmatrix} 0 & \delta^\alpha_\gamma \\ -C_{\gamma\rho} & 0 \end{bmatrix}\begin{bmatrix} 0 & \delta^\rho_\lambda \\ -C_{\lambda\beta} & 0 \end{bmatrix} + (\gamma\leftrightarrow\lambda) \\ &= -\begin{bmatrix} \delta^\alpha_\gamma C_{\lambda\beta}+\delta^\alpha_\lambda C_{\gamma\beta} & 0 \\ 0 & C_{\gamma\lambda}+C_{\lambda\gamma} \end{bmatrix}. \end{aligned} \tag{116}$$

The form of the lower diagonal piece immediately tells us that unless $C_{\gamma\lambda}$ is antisymmetric, the right hand side of (116) cannot be a linear combination of $\mathbf{J_a}$, $\mathbf{J_{ab}}$ and $\mathbf{Q}$. In that case, new bosonic generators with nonzero entries in this diagonal block will be required to close the algebra (and possibly more than one). This relation also shows that the upper diagonal block is a collection of matrices $\mathbf{M}_{\gamma\lambda}$ whose components take the form

$$(M_{\gamma\lambda})^\alpha_\beta = -(\delta^\alpha_\gamma C_{\lambda\beta}+\delta^\alpha_\lambda C_{\gamma\beta}).$$

Multiplying both sides of this relation by C, one finds

$$(CM_{\gamma\lambda})_{\alpha\beta} = -(C_{\alpha\gamma}C_{\lambda\beta}+C_{\alpha\lambda}C_{\gamma\beta}), \tag{117}$$

which is symmetric in $(\alpha\beta)$. This means that the bosonic generators can only include those matrices in the Dirac algebra such that, when multiplied by $\mathbf{C}$ on the left ($\mathbf{CI}$, $\mathbf{C\Gamma}_a$, $\mathbf{C\Gamma}_{a_1a_2}$, ..., $\mathbf{C\Gamma}_{a_1a_2\cdots a_D}$) turn out to be symmetric. The other consequence of this is that, if one wants to have the AdS algebra as part of the superalgebra, both $\mathbf{C\Gamma}_a$ and $\mathbf{C\Gamma}_{ab}$ should be symmetric matrices. Now, multiplying (114) by $\mathbf{C}$ from the right, we have

$$(\mathbf{C\Gamma}_a)^{\mathrm{T}} = \lambda\eta\mathbf{C\Gamma}_a, \tag{118}$$

which means that we need

$$\lambda\eta = 1. \tag{119}$$

From (114) and (115), it can be seen that

$$(\mathbf{C\Gamma}_{ab})^{\mathrm{T}} = -\lambda\mathbf{C\Gamma}_{ab},$$

which in turn requires

$$\lambda = -1 = \eta.$$

This means that $\mathbf{C}$ is antisymmetric ($\lambda=-1$) and then the lower diagonal block in (116) vanishes identically. However, the values of λ and η cannot be freely chosen but are fixed by the spacetime dimension as is shown in the following table

D	λ	η
3	-1	-1
5	-1	$+1$
7	$+1$	-1
9	$+1$	$+1$
11	-1	-1

and the pattern repeats mod 8 (see Ref.[62] for details). This table shows that the simple cases occur for dimensions 3 mod 8, while for the remaining cases life is a little harder. For $D = 7$ mod 8 the need to match the lower diagonal block with some generators can be satisfied quite naturally by including several spinors labelled with a new index, ψ_i^α, $i = 1, ..., \mathcal{N}$, and the generator of supersymmetry should also carry the same index. This means that there are actually $\mathcal{N}$ supercharges or, as it is usually said, the theory has an extended supersymmetry ($\mathcal{N} \geq \in$). For $D = 5$ mod 4 instead, the superalgebra can be made to close in spite of the fact that $\eta = +1$ if one allows complex spinor representations, which is a particular form of extended supersymmetry since now $\mathbf{Q}_\gamma$ and $\bar{\mathbf{Q}}^\gamma$ are independent.

So far we have only given some restrictions necessary to close the algebra so that the AdS generators appear in the anticommutator of two supercharges. AS we have observed, in general, apart from $\mathbf{J}_a$ and $\mathbf{J}_{ab}$, other matrices will occur in the r.h.s. of the anticommutator of $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ which extends the AdS algebra into a larger bosonic algebra. This happens even in the cases where there is no extended supersymmetry ($\mathcal{N}$=1).

The bottom line of this construction is that the supersymmetric extension of the AdS algebra for each odd dimension falls into one of these families:

$D = 3$ mod 8 (Majorana representation, $\mathcal{N} \geq 1$),

$D = 7$ mod 8 (Majorana representation, even $\mathcal{N}$), and

$D = 5$ mod 4 (complex representations, $\mathcal{N} \geq 1$ [or $2\mathcal{N}$ real spinors]).

The corresponding superalgebras[17]were computed by van Holten and Van Proeyen for $D = 2, 3, 4$ mod 8 in Ref. [55], and in the other cases, in Refs.[61, 62]:

D	S-Algebra	Conjugation Matrix
3 mod 8	$osp(m\|N)$	$C^T = -C$
7 mod 8	$osp(N\|m)$	$C^T = C$
5 mod 4	$usp(m\|N)$	$C^\dagger = C$

9 CS Supergravity Actions

In the previous sections we saw how to construct CS actions for the AdS connection for any $D = 2n + 1$. Now, we will see how to repeat this construction for the connection of a larger superalgebra in which AdS is embedded. Consider an arbitrary connection one-form $\mathbf{A}$, with values in some Lie algebra $\mathfrak{g}$, whose curvature is $\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$. Then, the $2n$-form

$$\mathcal{C}_{2n} \equiv < \mathbf{F} \wedge \cdots \wedge \mathbf{F} >, \tag{120}$$

[17]The algebra $osp(p|q)$ (resp. $usp(p|q)$) is that which generates the orthosymplectic (resp. unitary-symplectic) Lie group. This group is defined as the one that leaves invariant the quadratic form $G_{AB}z^A z^B = g_{ab}x^a x^b + \gamma_{\alpha\beta}\theta^\alpha\theta^\beta$, where g_{ab} is a p-dimensional symmetric (resp. hermitean) matrix and $\gamma_{\alpha\beta}$ is a q-dimensional antisymmetric (resp. anti-hermitean) matrix.

is invariant under the group whose Lie algebra is $\mathfrak{g}$, provided the bracket $< \cdots >$ is an invariant tensor of the group. Furthermore, $\mathcal{C}_{2n}$ is closed: $d\mathcal{C}_{2n} = 0$, and therefore can be locally written as an exact form,

$$\mathcal{C}_{2n} = d\mathbf{L}_{2n-1}.$$

The $(2n-1)$-form $\mathbf{L}_{2n-1}$ is a CS lagrangian, and therefore the problem reduces to finding the invariant bracket. The canonical –and in many cases unique– choice of invariant tensor with the features required here is the **supertrace**,

$$\langle \mathbf{F} \wedge \cdots \wedge \mathbf{F} \rangle =: STr\left[\Theta \mathbf{F} \wedge \cdots \wedge \mathbf{F}\right], \tag{121}$$

where Θ is an invariant matrix in the group, and the supertrace is defined as follows: if a matrix has the form

$$\mathbf{M} = \begin{bmatrix} N^a_b & F^a_\beta \\ H^\alpha_b & S^\alpha_\beta \end{bmatrix},$$

where a, b are (bosonic) tensor indices and α, β are (fermionic) spinor indices, then $STr[\mathbf{M}] = Tr[\mathbf{N}] - Tr[\mathbf{S}] = N^a_a - S^\alpha_\alpha$.

If we call $\mathbf{G}_M$ the generators of the Lie algebra, so that $\mathbf{A} = G_M A^M$, $\mathbf{F} = G_M F^M$, then

$$\begin{aligned} \mathcal{C}_{2n} &= STr\left[\Theta \mathbf{G}_{M_1} \cdots \mathbf{G}_{M_n}\right] F^{M_1} \cdots F^{M_n} \\ &= g_{\mathrm{M_1 \cdots M_n}} F^{M_1} \cdots F^{M} = d\mathbf{L}_{2n-1}, \end{aligned} \tag{122}$$

where $g_{\mathrm{M_1 \cdots M_n}}$ is an invariant tensor of rank n in the Lie algebra. Thus, the steps to construct the CS lagrangian are straightforward: Take the supertrace of all products of generators in the superalgebra and solve equation (122) for $\mathbf{L}_{2n-1}$. Since the superalgebras are different in each dimension, the CS lagrangians differ in field content and dynamical structure from one dimension to the next, although the invariance properties are similar in all cases. The action

$$I^{CS}_{2n-1}[\mathbf{A}] = \int \mathbf{L}_{2n-1} \tag{123}$$

is invariant, up to surface terms, under the local gauge transformation

$$\delta \mathbf{A} = \nabla \mathbf{\Lambda}, \tag{124}$$

where $\mathbf{\Lambda}$ is a zero-form with values in the Lie algebra $\mathfrak{g}$, and ∇ is the exterior covariant derivative in the representation of $\mathbf{A}$. In particular, under a supersymmetry transformation, $\mathbf{\Lambda} = \bar{\epsilon}^i Q_i - \bar{Q}^i \epsilon_i$, and

$$\delta_\epsilon \mathbf{A} = \begin{bmatrix} \epsilon^k \bar{\psi}_k - \psi^k \bar{\epsilon}_k & D\epsilon_j \\ -D\bar{\epsilon}^i & \bar{\epsilon}^i \psi_j - \bar{\psi}^i \epsilon_j \end{bmatrix}, \tag{125}$$

where D is the covariant derivative on the bosonic connection,

$$D\epsilon_j = \left(d + \frac{1}{2l} e^a \mathbf{\Gamma}_a + \frac{1}{4} \omega^{ab} \mathbf{\Gamma}_{ab} + \frac{1}{2r!} b^{[r]} \mathbf{\Gamma}_{[r]} \right) \epsilon_j - a^i_j \epsilon_i.$$

9.1 Examples of AdS-CS SUGRAs

Now we examine some simple examples of anti de Sitter Chern-Simons supergravities. For more detailed discussions and other examples, see [46][61][62][72].

A. D=3

The simplest locally supersymmetric extension of AdS occurs in three dimensions for $\mathcal{N}$=1. The lagrangian is [39]

$$L = \epsilon_{abc}\left[\frac{R^{ab}e^c}{l} + \frac{e^a e^b e^c}{3l^3}\right] + \bar{\psi}\overline{\nabla}\psi, \tag{126}$$

where $\overline{\nabla}$ stands for the AdS covariant derivative $\overline{\nabla} = d + \frac{1}{2l}e^a\Gamma_a + \frac{1}{4}\omega^{ab}\Gamma_{ab}$. It is straightforward to show that this action is invariant –up to surface terms– under supersymmetry,

$$\delta_\epsilon\psi = \overline{\nabla}\epsilon, \quad \delta_\epsilon e^a = \frac{-l}{4}\bar{\psi}\Gamma^a\epsilon, \quad \delta_\epsilon\omega^{ab} = \frac{1}{4}\bar{\psi}\Gamma^{ab}\epsilon.$$

The proof of invariance is direct and no field equations need to be invoked (off-shell local SUSY). Action (126) can also be written as

$$L = \langle \mathbf{A}d\mathbf{A} + \frac{2}{3}\mathbf{A}^3\rangle,$$

where $\mathbf{A}$ is the connection for the superalgebra $osp(2|1)$,

$$\mathbf{A} = \frac{1}{l}e^a\mathbf{J}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + \bar{\mathbf{Q}}\psi, \tag{127}$$

and $\langle\cdots\rangle$ stands for the supertrace of the matrix representation for $\mathbf{J}_a$, $\mathbf{J}_{ab}$ and $\mathbf{Q}$ defined in (121) (with an appropriate Θ insertion). The only nonvanishing brackets are:

$$\langle\mathbf{J}_{ab}\mathbf{J}_c\rangle = \langle\mathbf{J}_a\mathbf{J}_b\mathbf{J}_c\rangle = \frac{1}{2}\epsilon_{abc}, \tag{128}$$

$$\langle\mathbf{Q}\,\mathbf{Q}\rangle = C, \tag{129}$$

$$\langle\mathbf{Q}\,\mathbf{Q}\,\mathbf{J}_a\rangle = \frac{1}{4}C\Gamma_a, \tag{130}$$

$$\langle\mathbf{Q}\,\mathbf{Q}\,\mathbf{J}_{ab}\rangle = \frac{1}{4}C\Gamma_{ab}. \tag{131}$$

B. D=5

In this case the supergroup is $U(2,2|\mathcal{N})$. The associated connection can be written as

$$\mathbf{A} = e^a\mathbf{J}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + A^K\mathbf{T}_K + (\bar{\psi}^r\mathbf{Q}_r - \bar{\mathbf{Q}}^r\psi_r) + A\mathbf{Z}, \tag{132}$$

where the generators $\mathbf{J}_a$, $\mathbf{J}_{ab}$, form an AdS algebra ($so(4,2)$), $\mathbf{T}_K$ ($K = 1, \cdots \mathcal{N}^2 - 1$) are the generators of $su(\mathcal{N})$, $\mathbf{Z}$ generates a $U(1)$ subgroup and $\mathbf{Q}, \bar{\mathbf{Q}}$ are the supersymmetry generators, which transform in a vector representation of $SU(\mathcal{N})$. The Chern-Simons Lagrangian for this gauge algebra is defined by the relation $dL = iSTr[\mathbf{F}^3]$, where $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2$ is the (antihermitean) curvature. Using this definition, one obtains the lagrangian originally discussed by Chamseddine in [40],

$$L = L_G(\omega^{ab}, e^a) + L_{su(\mathcal{N})}(A^r_s) + L_{u(1)}(\omega^{ab}, e^a, A) + L_F(\omega^{ab}, e^a, A^r_s, A, \psi_r), \quad (133)$$

with

$$\begin{array}{rcl} L_G & = & \frac{1}{8}\epsilon_{abcde}\left[R^{ab}R^{cd}e^e/l + \frac{2}{3}R^{ab}e^c e^d e^e/l^3 + \frac{1}{5}e^a e^b e^c e^d e^e/l^5\right] \\ L_{su(\mathcal{N})} & = & -Tr\left[\mathbf{A}(d\mathbf{A})^2 + \frac{3}{2}\mathbf{A}^3 d\mathbf{A} + \frac{3}{5}\mathbf{A}^5\right] \\ L_{u(1)} & = & \left(\frac{1}{4^2} - \frac{1}{\mathcal{N}^2}\right)A(dA)^2 + \frac{3}{4l^2}\left[T^a T_a - R^{ab}e_a e_b - l^2 R^{ab}R_{ab}/2\right]A \;, \\ & & +\frac{3}{\mathcal{N}}F^r_s F^s_r A \\ L_f & = & \frac{3}{2i}\left[\bar{\psi}^r \mathcal{R}\nabla\psi_r + \bar{\psi}^s \mathcal{F}^r_s \nabla\psi_r\right] + c.c. \end{array} \quad (134)$$

where $A^r_s \equiv A^K(\mathbf{T}_K)^r_s$ is the $su(\mathcal{N})$ connection, F^r_s is its curvature, and the bosonic blocks of the supercurvature: $\mathcal{R} = \frac{1}{2}T^a\mathbf{\Gamma}_a + \frac{1}{4}(R^{ab} + e^a e^b)\mathbf{\Gamma}_{ab} + \frac{i}{4}dA\mathbf{I} - \frac{1}{2}\psi_s\bar{\psi}^s$, $\mathcal{F}^r_s = F^r_s + \frac{i}{\mathcal{N}}dA\delta^r_s - \frac{1}{2}\bar{\psi}^r\psi_s$. The cosmological constant is $-l^{-2}$, and the $U(2,2|\mathcal{N})$ covariant derivative ∇ acting on ψ_r is

$$\nabla\psi_r = D\psi_r + \frac{1}{2l}e^a_a\mathbf{\Gamma}\psi_r - A^s_r\psi_s + i\left(\frac{1}{4} - \frac{1}{\mathcal{N}}\right)A\psi_r. \quad (135)$$

where D is the covariant derivative in the Lorentz connection.

The above relation implies that the fermions carry a $u(1)$ "electric" charge given by $e = \left(\frac{1}{4} - \frac{1}{\mathcal{N}}\right)$. The purely gravitational part, L_G is equal to the standard Einstein-Hilbert action with cosmological constant, plus the dimensionally continued Euler density[18].

The action is by construction invariant –up to a surface term– under the local (gauge generated) supersymmetry transformations $\delta_\Lambda\mathbf{A} = -(d\mathbf{\Lambda} + [\mathbf{A}, \mathbf{\Lambda}])$ with $\mathbf{\Lambda} = \bar{\epsilon}^r\mathbf{Q}_r - \bar{\mathbf{Q}}^r\epsilon_r$, or

$$\begin{array}{rcl} \delta e^a & = & \frac{1}{2}\left(\bar{\epsilon}^r\mathbf{\Gamma}^a\psi_r - \bar{\psi}^r\mathbf{\Gamma}^a\epsilon_r\right) \\ \delta\omega^{ab} & = & -\frac{1}{4}\left(\bar{\epsilon}^r\mathbf{\Gamma}^{ab}\psi_r - \bar{\psi}^r\mathbf{\Gamma}^{ab}\epsilon_r\right) \\ \delta A^r_s & = & -i\left(\bar{\epsilon}^r\psi_s - \bar{\psi}^r\epsilon_s\right) \\ \delta\psi_r & = & -\nabla\epsilon_r \\ \delta\bar{\psi}^r & = & -\nabla\bar{\epsilon}^r \\ \delta A & = & -i\left(\bar{\epsilon}^r\psi_r - \bar{\psi}^r\epsilon_r\right). \end{array}$$

[18]The first term in L_G is the dimensional continuation of the Euler (or Gauss-Bonnet) density from two and four dimensions, exactly as the three-dimensional Einstein-Hilbert lagrangian is the continuation of the the two dimensional Euler density. This is the leading term in the limit of vanishing cosmological constant ($l \to \infty$), whose local supersymmetric extension yields a nontrivial extension of the Poincaré group [60].

As can be seen from (134) and (135), for $\mathcal{N}$=4 the $U(1)$ field A looses its kinetic term and decouples from the fermions (the gravitino becomes uncharged with respect to $U(1)$). The only remnant of the interaction with the A field is a dilaton-like coupling with the Pontryagin four forms for the AdS and $SU(\mathcal{N})$ groups (in the bosonic sector). As it is shown in Ref.[72], the case $\mathcal{N}$=4 is also special at the level of the algebra, which becomes the superalgebra $su(2,2|4)$ with a $u(1)$ central extension.

In the bosonic sector, for $\mathcal{N}$=4, the field equation obtained from the variation with respect to A states that the Pontryagin four form of AdS and $SU(\mathcal{N})$ groups are proportional. Consequently, if the spatial section has no boundary, the corresponding Chern numbers must be related. Since $\Pi_4(SU(4)) = 0$, the above implies that the Pontryagin plus the Nieh-Yan number must add up to zero.

C. D=11

In this case, the smallest AdS superalgebra is $osp(32|1)$ and the connection is

$$\mathbf{A} = e^a \mathbf{J}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + \frac{1}{5!}b^{abcde}\mathbf{J}_{abcde} + \bar{Q}\psi, \tag{136}$$

where b^{abcde} is a totally antisymmetric fifth-rank Lorentz tensor one-form. Now, in terms of the elementary bosonic and fermionic fields, the CS form in $\mathbf{L}_{2n-1}$reads

$$\mathbf{L}_{11}^{osp(32|1)}(\mathbf{A}) = L_{11}^{sp(32)}(\mathbf{\Omega}) + L_f(\mathbf{\Omega}, \psi), \tag{137}$$

where $\mathbf{\Omega} \equiv \frac{1}{2}(e^a\mathbf{\Gamma}_a + \frac{1}{2}\omega^{ab}\mathbf{\Gamma}_{ab} + \frac{1}{5!}b^{abcde}\mathbf{\Gamma}_{abcde})$ is an $sp(32)$ connection [61, 62, 73]. The bosonic part of (137) can be written as

$$L_{11}^{sp(32)}(\mathbf{\Omega}) = 2^{-6}L_{G\,11}^{AdS}(\omega, e) - \frac{1}{2}L_{T\,11}^{AdS}(\omega, e) + L_{11}^{b}(b, \omega, e),$$

where $L_{G\,11}^{AdS}$ is the CS form associated to the 12-dimensional Euler density, and $L_{T\,11}^{AdS}$ is the CS form whose exterior derivative is the Pontryagin form for $SO(10,2)$ in 12 dimensions. The fermionic lagrangian is

$$\begin{aligned} L_f \;\; = \;\; & 6(\bar{\psi}\mathcal{R}^4 D\psi) - 3\left[(D\bar{\psi}D\psi) + (\bar{\psi}\mathcal{R}\psi)\right](\bar{\psi}\mathcal{R}^2 D\psi) \\ & -3\left[(\bar{\psi}\mathcal{R}^3\psi) + (D\bar{\psi}\mathcal{R}^2 D\psi)\right](\bar{\psi}D\psi) + \\ & 2\left[(D\bar{\psi}D\psi)^2 + (\bar{\psi}\mathcal{R}\psi)^2 + (\bar{\psi}\mathcal{R}\psi)(D\bar{\psi}D\psi)\right](\bar{\psi}D\psi), \end{aligned}$$

where $\mathcal{R} = d\mathbf{\Omega} + \mathbf{\Omega}^2$ is the $sp(32)$ curvature. The supersymmetry transformations (125) read

$$\delta e^a = \tfrac{1}{8}\bar{\epsilon}\mathbf{\Gamma}^a\psi \qquad \delta\omega^{ab} = -\tfrac{1}{8}\bar{\epsilon}\mathbf{\Gamma}^{ab}\psi$$

$$\delta\psi = D\epsilon \qquad \delta b^{abcde} = \tfrac{1}{8}\bar{\epsilon}\mathbf{\Gamma}^{abcde}\psi.$$

Standard (CJS) eleven-dimensional supergravity [69] is an $\mathcal{N}$=1 supersymmetric extension of Einstein-Hilbert gravity that cannot admit a cosmological constant [70, 71]. An $\mathcal{N}>1$ extension of the CJS theory is not known. In our case, the cosmological constant is necessarily nonzero by construction and the extension simply requires including an internal $so(\mathcal{N})$ gauge field coupled to the fermions. The resulting lagrangian is an $osp(32|\mathcal{N})$ CS form [73].

9.2 Wigner-Inönüu Contractions

The Poincaré group is the symmetry of the spacetime that best approximates the world around us at low energy, Minkowski space. The Poincaré group can be viewed as the limit of vanishing cosmological constant or infinite radius ($\Lambda \sim \pm l^{-2} \to 0$) of the de Sitter or anti-de Sitter group. This deformation is called a Wigner-Inönüu (**WI**) contraction of the group that can be implemented in the algebra through a rescaling the generators: $\mathbf{J}_a \to \mathbf{P}_a = l^{-1}\mathbf{J}_a$, $\mathbf{J}_{ab} \to \mathbf{J}_{ab}$. Thus, starting from the AdS symmetry in 3+1 dimensions ($SO(3,2)$), the rescaled algebra is

$$\begin{aligned} [\mathbf{P}_a, \mathbf{P}_b] &= l^{-2}\mathbf{J}_{ab} \\ [\mathbf{J}_{ab}, \mathbf{P}_c] &= \mathbf{P}_a\eta_{bc} - \mathbf{P}_b\eta_{ac} \\ [\mathbf{J}_{ab}, \mathbf{J}_{cd}] &\sim \mathbf{J}_{ad}\eta_{bc} - \cdots, \end{aligned} \tag{138}$$

and therefore, in the limit $l \to \infty$, $\mathbf{P}_a$ becomes a generator of translations. The resulting contraction is the Poincaré group, $SO(3,2) \to ISO(3,1)$. A similar contraction takes the de Sitter group into Poincaré, or in general, $SO(p,q) \to ISO(p,q-1)$.

In general, the WI contractions change the structure constants and the Killing metric of the algebra without changing the number of generators, but the resulting algebra is still a Lie algebra. Since some structure constants may go to zero under by contraction, some generators become commuting and end up forming an abelian subalgebra. So, the contraction of a semisimple algebra is not necessarily semisimple, like in the above example. For a detailed discussion of contractions, see, *e.g.*, [74], and for a nice historical note see [75]. As could be expected, the contraction of a group induces a contraction of representations and therefore it is possible to obtain a lagrangian for the contracted group by a corresponding limiting procedure. However, as it was immediately noticed by its inventors, the WI contractions can give rise to unfaithful representations. In other words, the limit representation may not be an irreducible faithful representation of the contracted group. Therefore, the procedure to obtain an action for the contracted group is far from a straightforward operation of taking the limit of the original action. This is particularly difficult in the case of the supersymmetric actions. In fact, the actions for the supersymmetric extension of the Poincaré group was obtained in [60], and then the Chern-Simons actions for the SUSY extensions of AdS were found in [61], and it is puzzling that the naive limit of the latter do not in general reproduce the former. However, this should not be surprising in the light of the previous discussion.

9.3 Minimal Super-Poincaré Theory

In [60], a general form of the CS lagrangian for the minimal SUSY extension of the Poincaré algebra was constructed. In $2+1$ dimensions the local symmetry group is super-Poincaré, whose algebra includes the Poincaré generators and one Majorana supercharge Q. For $D=5$ the supercharge is complex (Dirac) spinor and the algebra also acquires a central extension (one generator which commutes with the rest of the algebra). In general, the algebra consists of the Poincaré generators, the supercharge Q (and its adjoint $\bar{Q}$), and a fifth rank antisymmetric Lorentz tensor $\mathbf{Z}_{abcdf}$. Clearly, for $D=3,5$ the general case reduces to the cases mentioned above. The connection is

$$\mathbf{A} = e^a\mathbf{P}_a + \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + \frac{1}{5!}b^{abcde}\mathbf{Z}_{abcde} + \bar{\mathbf{Q}}\psi - \bar{\psi}\mathbf{Q}, \tag{139}$$

where $\mathbf{P}_a$ and $\mathbf{J}_{ab}$ are the generators of the Poincaré group, Z_{abcdf} is a fifth rank Lorentz tensor which commutes with $\mathbf{P}_a$ and Q, and

$$\{\mathbf{Q}^\alpha, \bar{\mathbf{Q}}_\beta\} = -i(\Gamma^a)^\alpha_\beta\mathbf{P}_a - i(\Gamma^{abcde})^\alpha_\beta\mathbf{Z}_{abcde} \tag{140}$$

In the dimensions in which there exists a Majorana representation, $\bar{\mathbf{Q}} = C\mathbf{Q}$, the number of fermionic generators can be reduced in half.

The action has three terms,

$$I_{2n+1} = I_G + I_b + I_\psi. \tag{141}$$

The first term describes locally Poincaré-invariant gravity,

$$I_G[e,\omega] = \int \epsilon_{a_1\cdots a_{2n+1}}R^{a_1a_2}\cdots R^{a_{2n-1}a_{2n}}e^{a_{2n+1}}. \tag{142}$$

The second represents the coupling between the fifth-rank Lorentz tensor 1-form $\mathbf{b}^{abcde}$ and gravity,

$$I_b = -\frac{1}{6}\int R_{abc}R_{de}b^{abcde}, \tag{143}$$

and the third includes the fermionic 1-form (gravitino)

$$I_\psi = \frac{i}{6}\int R_{abc}[\bar{\psi}\Gamma^{abc}D\psi + D\bar{\psi}\Gamma^{abc}\psi]. \tag{144}$$

Here we have defined the symbol R_{abc} as

$$R_{abc} := \epsilon_{abca_1\cdots a_{D-3}}R^{a_1a_2}\cdots R^{a_{D-4}a_{D-3}} \tag{145}$$

This action is invariant under Lorentz rotations,

$$\begin{aligned}\delta\omega^{ab} &= -D\lambda^{ab},\;\; \delta e^{a}=\lambda^{a}_{\ b}e^{b},\;\; \delta\psi=\frac{1}{4}\lambda^{ab}\Gamma_{ab}\psi, \\ \delta b^{abcde} &= \lambda^{a}_{\ a'}b^{a'bcde}+\lambda^{b}_{\ b'}b^{ab'cde}+...+\lambda^{e}_{\ e'}b^{abcde'}\end{aligned} \tag{146}$$

Poincaré translations,

$$\delta\omega^{ab}=0,\;\; \delta e^{a}=-D\lambda^{a},\;\; \delta\psi=0,\;\; \delta b^{abcde}=0, \tag{147}$$

and local SUSY transformations

$$\delta\omega^{ab}=0,\;\; \delta e^{a}=\frac{1}{2}\bar{\epsilon}\Gamma^{a}\psi,\;\; \delta\psi=D\epsilon,\;\; \delta b^{abcde}=0. \tag{148}$$

The bracket $\langle\cdots\rangle$ that gives rise to the action (142-144) in the general form is given by

$$\langle \mathbf{J}_{a_1a_2}\cdots\mathbf{J}_{a_{D-2}a_{D-1}}\mathbf{P}_{a_D}\rangle=\epsilon_{a_1a_2\cdots a_D} \tag{149}$$

$$\langle \mathbf{J}_{a_1a_2}\cdots\mathbf{J}_{a_{D-4}a_{D-3}}\mathbf{J}_{fg}\mathbf{Z}_{abcde}\rangle=-\epsilon_{a_1\cdots a_{D-3}abc}\eta_{fd}\eta_{ge}\pm\text{(permutations)} \tag{150}$$

$$\langle \mathbf{Q}\mathbf{J}_{a_1a_2}\cdots\mathbf{J}_{a_{D-4}a_{D-3}}\bar{\mathbf{Q}}\rangle=1i^{n}\Gamma_{a_1\cdots a_{D-3}}. \tag{151}$$

In the eleven-dimensional case, one can envision this algebra as resulting from a WI contraction of the $osp(32|1)$ superalgebra with connection (136). In fact, it seems natural to expect that if in the limit $l\rightarrow\infty$, the generators $\mathbf{P}_a=l^{-1}\mathbf{J}_a$, $\mathbf{Z}_{abcde}=l^{-1}\mathbf{J}_{abcde}$, $\mathbf{Q}'=l^{-1/2}\mathbf{Q}$, satisfy the minimal Poincaré algebra with connection (139) (if $\bar{\mathbf{Q}}=C\mathbf{Q}$). However, is not straightforward how to perform the limit in the lagrangian and some field redefinitions are needed in order to make contact between the two theories.

9.4 M-Algebra Extension of the Poincaré Group

The CS actions discussed previously have been constructed looking for the representation that extends the bosonic fields e^a and ω^{ab} completing the SUSY multiplet. This is not as elegant as an algebraic construction from first principles in which one only has the input of an abstract algebra, but has the advantage that the lagrangian is determined at once for the relevant fields. An additional difficulty of a formal approach is that it requires knowing the invariant tensors of the algebra which, in the explicit representation takes the form of the (super)trace $\langle\cdots\rangle$.

In eleven dimensions there is, besides the Poincaré SUGRA of the previous section, a new extension that corresponds to a supersymmetry algebra with more bosonic generators [76]. The algebra includes, apart from the Poincaré generators $\mathbf{J}_{ab}$ and $\mathbf{P}_a$, a Majorana supercharge $\mathbf{Q}_\alpha$ and two additional bosonic generators that close the supersymmetry algebra,

$$\{\mathbf{Q}_\alpha,\mathbf{Q}_\beta\}=(C\Gamma^{a})_{\alpha\beta}\,\mathbf{P}_a+(C\Gamma^{ab})_{\alpha\beta}\mathbf{Z}_{ab}+(C\Gamma^{abcde})_{\alpha\beta}\mathbf{Z}_{abcde}\;. \tag{152}$$

where the charge conjugation matrix C is antisymmetric. The "central charges" $\mathbf{Z}_{ab}$ and $\mathbf{Z}_{abcde}$ are tensors under Lorentz rotations but are otherwise Abelian generators. The algebra (152) is known as M-algebra because it is the expected gauge invariance of M Theory [77].

The M-algebra has more generators than the minimal super Poincaré algebra of the previous section, because the new $\mathbf{Z}_{ab}$ has no match there. Thus, the M-algebra has $\begin{pmatrix} 11 \\ 2 \end{pmatrix} = 55$ more generators than both the minimal super Poincaré and the $osp(32|1)$ algebras. This means, in particular, that the M-algebra cannot be found by a WI contraction from either one of these two algebras.

The action is a supersymmetric extension of the Poincaré invariant gravitational action (142) in eleven dimensions,

$$\begin{aligned} I_\alpha &= I_G + I_\psi - \frac{\alpha}{6}\int_{M_{11}} R_{abc}R_{de}b^{abcde} \\ &+ 8(1-\alpha)\int_{M_{11}} [R^2 R_{ab} - 6(R^3)_{ab}]R_{cd}\left(\bar{\psi}\Gamma^{abcd}D\psi - 12R^{[ab}b^{cd]}\right), \end{aligned} \tag{153}$$

where I_G and I_ψ are

$$I_G[e,\omega] = \int_{M_{11}} \epsilon_{a_1\cdots a_{11}} R^{a_1a_2}\cdots R^{a_9a_{10}}e^{a_{11}}, \tag{154}$$

$$I_\psi = -\frac{1}{3}\int_{M_{11}} R_{abc}\bar{\psi}\Gamma^{abc}D\psi, \tag{155}$$

and $R^2 := R^{ab}R_{ba}$ and $(R^3)^{ab} := R^{ac}R_{cd}R^{db}$. Here α is an arbitrary dimensionless constant whose meaning will be discussed below. The action is invariant under local supersymmetry transformations obtained from a gauge transformation of the M-connection (157) with parameter $\lambda = \epsilon^\alpha \mathbf{Q}_\alpha$,

$$\begin{array}{ll} \delta_\varepsilon e^a = \bar{\epsilon}\Gamma^a\psi, & \delta_\varepsilon\psi = D\epsilon, \quad \delta_\varepsilon\omega^{ab} = 0, \\ \delta_\varepsilon b^{ab} = \bar{\epsilon}\Gamma^{ab}\psi, & \delta_\varepsilon b^{abcde} = \bar{\epsilon}\Gamma^{abcde}\psi. \end{array} \tag{156}$$

The field content is given by the components of the M-algebra connection,

$$\mathbf{A} = \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + e^a\mathbf{P}_a + \psi^\alpha\mathbf{Q}_\alpha + b^{ab}\mathbf{Z}_{ab} + b^{abcde}\mathbf{Z}_{abcde}\ , \tag{157}$$

and the action can be written in terms of the bracket $\langle\cdots\rangle$ whose only nonzero entries are

$$\begin{aligned} &\langle J_{a_1a_2},\cdots,J_{a_9a_{10}},P_{a_{11}}\rangle = \tfrac{16}{3}\epsilon_{a_1\cdots a_{11}}\ , \\ &\langle J_{a_1a_2},\cdots,J_{a_9a_{10}},Z_{abcde}\rangle = -\alpha\tfrac{4}{9}\epsilon_{a_1\cdots a_8abc}\eta_{[a_9a_{10}][de]}\ , \\ &\left\langle J_{a_1a_2},J_{a_3a_4},J_{a_5a_6},J^{a_7a_8},J^{a_9a_{10}},Z^{ab}\right\rangle = (1-\alpha)\tfrac{16}{3}\left[\delta^{a_7\cdots a_{10}ab}_{a_1\cdots\cdots a_6} - \delta^{a_9a_{10}ab}_{a_1\cdots a_4}\delta^{a_7a_8}_{a_5a_6}\right] \\ &\langle Q,J_{a_1a_2},J^{a_3a_4},J^{a_5a_6},J^{a_7a_8},Q\rangle = \tfrac{32}{15}\left[C\Gamma_{a_1a_2}{}^{a_3\cdots a_8} + \right. \\ &\qquad\qquad \left.(1-\alpha)\left(3\delta^{a_3\cdots a_6}_{a_1a_2ab}C\Gamma^{a_7a_8ab} + 2C\Gamma^{a_3\cdots a_6}\delta^{a_7a_8}_{a_1a_2}\right)\right], \end{aligned}$$

where (anti-)symmetrization under permutations of each pair of generators is understood when all the indices are lowered.

It is natural to ask whether there is a link between this theory and the vanishing cosmological constant of $osp(32|1)$. As already mentioned, these theories cannot be related through a WI contraction, because contractions cannot increase the number of generators in the algebra. In fact the M-algebra can be obtained from the AdS algebra by a more general singular transformation called a **deformation** [78]. These deformations are analytic mappings in the algebra with the only restriction that they respect the Maurer-Cartan structure equations. They have been studied in Refs. [79], and more recently in [80].

The fact that this theory contains the free parameter α means that there is more than one way of deforming $osp(32|1)$ which produces an action supersymmetric under the transformations (156). For $\alpha = 1$ (153) reduces to the minimal Poincaré action (142-144). This means that the combination $I_{\alpha=0} - I_{\alpha=1}$ is supersymmetric by itself (although it does not describe a gravitational theory as it does not involve the vielbein).

9.5 Field Equations

The existence of this bracket allows writing the field equations for an CS theory in $2n+1$ dimensions in a manifestly covariant form as

$$\langle F^n G_A \rangle = 0. \tag{158}$$

In addition, if the $(2n+1)$-dimensional spacetime is conceived as the boundary of a $(2n+2)$-dimensional manifold, $\partial\Omega_{2n+2} = M_{2n+1}$, the CS action can also be written as $I = \int_{\Omega_{2n+2}} \langle F^{n+1} \rangle$, which describes a topological theory in $2n+2$ dimensions. In spite of its topological origin, the action does possess propagating degrees of freedom in $2n+1$ dimensions and hence it should not be thought of as a topological field theory.

The field equations are nonlinear for $D \geq 5$ and posses a rich dynamical structure due to the possibility of having zeros of different orders in (158). The perturbative field theories that can be constructed around each of these classical configurations have different particle content and correspond to different disconnected phases of the theory [81]. In particular, these phases can be seen as different vacua corresponding to different dimensional reductions of the starting theory [76].

9.6 Overview

Let us pause for a moment to recap what we have found so far. The supergravities presented here have two distinctive features: The fundamental field is always the connection $\mathbf{A}$ and, in their simplest form, they are pure CS systems (matter couplings are briefly discussed below). As a result, these theories possess a larger gravitational sector, including propagating spin connection. Contrary to what one could expect, the geometrical interpretation is quite clear, the field structure is simple and, in contrast with the standard cases, the supersymmetry transformations close off shell without auxiliary fields. The price to pay is to have a complex classical dynamics and a richer perturbative spectrum that changes from one background to another.

Field content and extensions with $\mathcal{N} > 1$

The field content of AdS CS Supergravities and the standard supergravities in $D = 5, 7, 11$ is compared in the following table:

D	Standard supergravity	CS supergravity	Algebra
5	$e^a_\mu\ \psi^\alpha_\mu\ \bar{\psi}_{\alpha\mu}$	$e^a_\mu\ \omega^{ab}_\mu\ A_\mu\ A^i_{j\mu}\ \psi^\alpha_{i\mu}\ \bar{\psi}^i_{\alpha\mu},\ i,j = 1, ...\mathcal{N}$	$usp(2,2\vert\mathcal{N})$
7	$e^a_\mu\ A_{[3]}\ a^i_{\mu j}\ \lambda^\alpha\ \phi\ \psi^{\alpha i}_\mu$	$e^a_\mu\ \omega^{ab}_\mu\ A^i_{\mu j}\ \psi^{\alpha i}_\mu,\ i,j = 1, ...\mathcal{N}$ =2n	$osp(\mathcal{N}\vert 8)$
11	$e^a_\mu\ A_{[3]}\ \psi^\alpha_\mu$	$e^a_\mu\ \omega^{ab}_\mu\ b^{abcde}_\mu\ \psi^\alpha_\mu\ ,\ i,j = 1, ...\mathcal{N}$	$osp(32\vert\mathcal{N})$

Standard supergravity in five dimensions is dramatically different from the theory presented here: apart from the graviton (e^a) and the complex gravitino (ψ_μ), there is a propagating spin connection and at least a $U(1)$ gauge field (A_μ) in the AdS theory which have no match in standard $D = 5$ SUGRA.

The standard $D = 7$ supergravity is an $\mathcal{N}$=2 theory (its maximal extension is $\mathcal{N}$=4), whose gravitational sector is given by Einstein-Hilbert gravity with cosmological constant and with a background invariant under $OSp(2|8)$ [82, 83]. Standard eleven-dimensional supergravity [69] is an $\mathcal{N}$=1 supersymmetric extension of Einstein-Hilbert gravity with vanishing cosmological constant. An $\mathcal{N}$ >1 extension of this theory is not known.

In our construction, the extensions to larger $\mathcal{N}$ are straightforward in any dimension. In $D = 7$, the index i is allowed to run from 2 to $2s$, and the lagrangian is a CS form for $osp(2s|8)$. In $D = 11$, one must include an internal $so(\mathcal{N})$ field and the lagrangian is an $osp(32|\mathcal{N})$ CS form [61, 62]. The cosmological constant is necessarily nonzero in all cases.

Gravity sector.

A most remarkable result from imposing the supersymmetric extension, is the fact that if one sets all fields, except those that describe the geometry $-e^a$ and $\omega^{ab}-$ to zero, the remaining action has no free parameters. This means that the gravity sector is uniquely fixed. This is remarkable because as we saw already for $D = 3$ and $D = 7$, there are several CS actions that one can construct for the AdS gauge group, the Euler CS form and the so-called exotic ones, that include torsion explicitly, and the coefficients for these different CS lagrangians is not determined by the symmetry considerations. So, even from a purely gravitational point of view, if the theory admits a supersymmetric extension, it has more predictive power than if it does not.

Relation with standard SUGRA

In all these CS theories ω^{ab} is an independent dynamical field, something that is conspicuously absent in standard SUGRA. The spin connection can be frozen out by imposing the torsion condition (the field equation obtained varying the action with respect to ω^{ab}, which determines T^a). In a generic –and sufficiently simple background–, this is an algebraic equation for ω^{ab}. As we mentioned in Section 5.2, substituting the solution $\omega^{ab} = \omega^{ab}(e, \cdots)$ gives a classically equivalent action principle in the reduced phase space.

Standard SUGRAs have a gravity sector described by the Einstei-Hilbert action. However, there is no simple limit that one can take in the Lovelock action to yield the EH action. The limit of vanishing cosmological constant gives a lagrangian which has $(D-1)/2$ powers of curvature, so that additionally, one should look at the action around a special configuration where the fields behave in a way that resembles the linearized excitations in EH gravity.

Certainly, additional field identifications should be made, some of them quite natural, like the identification of the totally antisymmetric part of ω_μ^{ab} in a coordinate basis, $k_{\mu\nu\lambda}$ (known as the contorsion tensor), with an abelian 3-form, $A_{[3]}$. (In 11 dimensions, one could also identify the totally antisymmetrized part of A_μ^{abcde} with an abelian 6-form $A_{[6]}$, whose exterior derivative, $dA_{[6]}$, is the dual of $F_{[4]} = dA_{[3]}$. Hence, in $D = 11$ the CS theory may contain the standard supergravity as well as some kind of dual version of it.)

In trying to make contact with standard SUGRA, the gauge invariance of the CS theory in its original form would be completely entangled by the elimination of the spin connection. Moreover, the identification between the tangent space and the base manifold brought in by the elimination of the vielbein in favor of the metric tensor destroys any hope to interpret the resulting action as a gauge theory with fiber bundle structure, but this is more or less the situation in standard SUGRAs where a part of the gauge invariance is replaced by invariance under diffeomorphisms.

Thus, the relation between the AdS CS SUGRAs and the standard SUGRAs is at best indirect and possibly only in one sector of the theory. One may ask, therefore, why is it necessary to show the existence of a connection between these two theories? The reason is historical (standard SUGRAs where here first) and of consistency (standard SUGRAs are known to be rather unique). So, even if it is a difficult and possibly unnecessary exercise, it is our responsibility to show that the connection exists [78].

Classical solutions

The field equations for these theories, in terms of the Lorentz components (ω, e, A, $\mathbf{A}$, ψ), are the different Lorentz tensor components for $< \mathbf{F}^{n-1}\mathbf{G}_M >= 0$. These equations have a very complex space of solutions, with different sectors with radically different dynamical behaviors.

It can be easily verified that in all these theories the anti-de Sitter space and with $\psi = b = \mathbf{A} = 0$, is a classical solution which corresponds to the most symmetric vacuum: all curvature components vanish, it has maximal set of Killing vectors, and since it is invariant under supersymmetry, it has also a maximal set of Killing spinors. This BPS state cannot decay into anything and what is most intriguing, it has no perturbations around it. In this sense, this is not a true vacuum since it cannot be populated with excitations like the vacuum in a quantum field theory. This "ultra vacuum" is a single topological state and the action around it is effectively a surface term, so it actually describes a field theory at the boundary: the action in this sector describes a conformal field theory at the boundary with the same gauge SUSY as the theory in the bulk.

There exist other less symmetric solutions which do allow perturbations around them, and these are more interesting states to look at. For instance, in the pure gravity (matter-free) sector, there exist spherically symmetric, asymptotically AdS standard [42], as well as topological black holes [84]. In the extreme case these black holes can be shown to be BPS states [85].

Spectrum

It may be impossible to establish a complete classification of all classical of the CS equations. The quantization of these systems is at the moment an open problem, the main obstacle being the complex vacuum structure, and the lack of perturbative expansion around many of them.

Some simple classical solutions are product spaces where one factor is maximally symmetric or constant curvature subspaces. A recurrent feature is that when one of the factors has vanishing AdS curvature, the other factor has indeterminate local geometry. This is because the field equations are typically a product of curvature two-forms equal to zero; therefore, if one factor vanishes, the others are not determined by the field equations. The components which are indeterminate are analogous to gauge degrees of freedom, and it can be seen that the theory actually has fewer degrees of freedom around these configurations. A dramatic example of this has been found in the CS supergravity for the M-Theory algebra, where a configuration was found in the 11-dimensional theory that corresponds to the spectrum of 4-dimensional de Sitter space, plus matter couplings [76].

The stability and positivity of the energy for the solutions of these theories is a highly nontrivial problem. As shown in Ref. [66], the number of degrees of freedom of bosonic CS systems for $D \geq 5$ is not constant throughout phase space and different regions can have radically different dynamical content. However, in a region where the rank of the symplectic form is maximal the theory may behave as a normal gauge system, and this condition would be stable under perturbations. As shown in [72] for $D = 5$, there exists a nontrivial central extension of the AdS superalgebra in anti-de Sitter space with a nontrivial $U(1)$ connection but no other matter fields. In this background the symplectic form has maximal rank and the gauge superalgebra is realized in the Dirac brackets.

This fact ensures a lower bound for the mass as a function of the other bosonic charges [86].

Representations

The mismatch between fermionic and bosonic states is most puzzling for those accustomed to standard supersymmetry, where the Fermi-Bose matching is such a central feature that has been used as synonym of SUSY. In fact, it is common to read that the signal expected to emerge from accelerator experiments at very high energy (above the supposed SUSY-breaking scale) is a pairing of states with equal mass, electric charge, parity, lepton or baryon number, etc., but with different fermion number. In the theories described here, no such signal should be expected, which may be a relief after so many years of fruitless search in this direction.

Another technical aspect related to the representation used in CS SUGRAs, is the avoidance of Fierz rearrangements (**FR**) which are a source of much suffering in standard SUSY and SUGRA. The FR are needed necessary to express bilinear products of spinors in terms of all possible products of Dirac matrices. Since in CS theories one only deals with exterior (wedge) products, only antisymmetrized products of Dirac matrices appear in the algebra. Also, exterior products of spinors always give irreducible representations, so it is not necessary to decompose these products in smaller irreps, as it happens in standard SUSY.

Matter couplings

It is possible to introduce minimal couplings to matter of the form $\mathbf{A} \cdot \mathbf{J}^{ext}$. For $D = 5$, the theory couples to an electrically charged $U(1)$ 0 brane (point charge), to $SU(4)$ -colored 0 branes (quarks) or to uncharged 2-brane, whose respective worldhistories couple to A_μ, $\mathbf{A}_\mu^{rs}$ and ω_μ^{ab} respectively. For $D = 11$, the theory admits a 5-brane and a 2-brane minimally coupled to b_μ^{abcde} and ω_μ^{ab} respectively.

Dynamical Content

The physical meaning of a theory is defined by the dynamics it displays both at the classical and quantum levels. In order to understand the dynamical contents of the classical theory, the physical degrees of freedom and their evolution equations must be identified. In particular, it should be possible –at least in principle– to separate the propagating modes from the gauge degrees of freedom, and from those which do not evolve independently at all (second class constraints). The standard way to do this is Dirac's constrained Hamiltonian analysis and has been applied to CS systems in [66]. In the Appendix, that analysis is summarized. It is however, fair to say that a number of open problems remain and it is a area of research which is at a very different stage of development compared with the previous discussion.

10 Final Comments

1. Everything we know about the gravitational interaction at the classical level, is described by Einstein's theory in four dimensions, which in turn is supported by a handful of experimental observations. There are many indications, however, that make it plausible to accept that our spacetime has more dimensions than those that meet the eye. In a spacetime of more than four dimensions, it is not logically necessary to consider the Einstein-Hilbert action as the best description for gravity. In fact, string theory suggests a Lovelock type action as more natural option [32]. The large number of free parameters in the Lovelock action, however, cannot be fixed by arguments from string theory. As we have shown, the only case in which there is a simple symmetry principle to fix these coefficients is odd dimensions and that leads to the Chern-Simons theories.

2. The CS theories have profound geometrical roots connecting them to topological invariants –the Euler and the Chern or Pontryagin classes. In the context of gravity, they appear in a very natural way in a framework where the affine and metric structures of the geometry are taken to be independent dynamical objects. If one demands furthermore the theory to admit supersymmetry, there is, in each dimensions essentially a unique extension which completely fixes the gravitational sector, including the precise role of torsion in the action.

3. The CS theories of gravity discussed in the first part of these notes possess nontrivial black hole solutions [43] which asymptotically approach spacetimes of constant negative curvature (AdS spacetimes). These solutions have a thermodynamical behavior which is unique among all possible black holes in competing Lovelock theories with the same asymptotics [47]. The specific heat of these black holes is positive and therefore they can always reach thermal equilibrium with their surroundings. These theories also admit solutions which represent black objects of other topologies, whose singularity is shrouded by horizons of non spherical topology [49]. Furthermore, these solutions seem to have well defined, quantum mechanically stable, BPS ground states [85].

4. The higher-dimensional Chern-Simons systems remain somewhat mysterious, especially in view of the difficulties to treat them as quantum theories. However, they have many ingredients that make CS theories likely quantum systems: They carry no dimensionful couplings, the only parameters they have are quantized, they are unique within the Lovelock family give rise to black holes with positive specific heat [47] and hence, stable against thermal fluctuations. Efforts to quantize CS systems seem promising at least in the cases in which the space admits a complex structure so that the symplectic form is a Kähler form [87].

5. It is too soon to tell whether string theory is the correct description of all interactions and constituents of nature. If it is the right scenario, and gravity is just a low energy effective theory, there would be a compelling reason to study gravity in higher dimensions, not just as an academic exercise as could have seemed in the time of Lanczos, but as a tool to study big bang cosmology or black hole physics. The truth is that a field theory can tell us a lot a bout the low energy phenomenology, in the same way that ordinary quantum mechanics tells us a lot about atomic physics even if we know that is all somehow contained in QED.

6. If the string scenario fails to deliver its promise, more work will be needed still to understand the field theories it is supposed to represent, in order to decipher their deeper interrelations. In any case, geometry is likely to be an important clue, very much in the same way that it is an essential element in Yang Mills and Einstein's theory. One can see the construction discussed in these lectures as a walking tour in this direction.

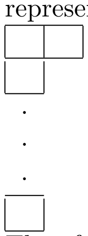
7. We can summarize the general features of Chern-Simons Theories in the following (incomplete) list:

• Truly gauge invariant theories. The dynamical field is a connection 1-form, their symmetry algebra closes off-shell, they have no dimensionful coupling constants.

• Gravity naturally included . They are fully covariant under general coordinate transformations. The vielbein and the spin connection can be combined into a connection for the (A)dS group or the Poincaré group.

• No derivatives higher than second order. As the formulation is in terms of exterior forms (without the Hodge *-dual) the lagrangian has at most first derivatives of the fields, and the field equations are also first order. If the torsion condition is used to eliminate the spin connection, the field equations become at most second order for the metric.

• No spins higher than 2. All component fields in the connection carry only one spacetime index (they are 1-forms), and they are antisymmetric tensors of arbitrary rank under the Lorentz group ($\sim b_\mu^{ab\cdots}$). Thus, they belong to representations of the rotation group whose Young tableaux are of the form:



Therefore, at most one symmetric tensor of rank two (spin 2) can be constructed with these fields contracting the spacetime index with one of the Lorentz indices, using the vielbein: $e_{a\mu}b_{\nu}^{ab\cdots} =: b_{\mu\nu}^{b\cdots}$ and therefore no fundamental fields of spin higher than two can be represented in these theories.

• No matching between fermions and bosons. Since the connection is not in the fundamental (vector) representation, but in the adjoint, and the spacetime transformations do not commute with the spinor generators, there should be no matching between fermionic and bosonic states. This shows that it is perfectly possible to have supersymmetry and yet have no supersymmetric partners for each known particle (sleptons, squarks, gluinos, etc.).

• Degenerate classical dynamics for $D \geq 5$ and trivial dynamics for $D \leq 3$. In general for $D = 2n + 1$, CS theories have field equations which are polynomials of degree n in curvatures. For $D \geq 5$, this gives rise to degeneracies in the symplectic structure with a corresponding breakdown in the dynamical evolution of the initial data. This gives produces a splitting of the phase space into many different phases with different degrees of freedom described by the same theory. For $D = 3$, instead, the local dynamics is trivial in the sense that there are no propagating degrees of freedom. This doesn't necessarily mean that the $D = 3$ is completely uninteresting.

• Dimensional reduction singles out $D = 4$ uncompactified dimensions. Regardless of at what dimension one may start, gravity has locally propagating perturbations (gravitons) in a product spacetime in which one of the factors is four-dimensional $D = 4$. This may have something to do with the observation that we live in a three-dimensional space.

Although many open questions remain to be addressed before CS theories can be used to describe the microscopic world, I hope to have convinced the reader that these are beautiful mathematical structures and rich physical systems worth studying. It would be really a shame if God didn't take advantage of so many interesting features somewhere in its creation.

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A Appendix: General dynamics of CS theories

Chern-Simons theories are exceptions to almost any feature of a standard field theory. This is because they are extremely singular dynamical systems:

• They are **constrained systems** like any gauge theory and therefore they have a non invertible Legendre transformation. This reflects the usual feature that some of the coordinates are non propagating gauge degrees of freedom and the corresponding constraints are the generators of gauge transformations.

• They have a **degenerate** symplectic matrix whose rank is not constant throughout phase space. This means that the separation between coordinates and momenta cannot be made uniformly throughout phase space, and there are regions where the number of degrees of freedom of the theory changes abruptly: **degenerate surfaces**.

• Their constraints are **irregular** in the sense that the number of functionally independent constraints is not constant throughout phase space. This issue is independent from the previous one, although not totally independent from it. The **irregularity surfaces** where the constraints fail to be independent define systems with dynamically different behavior.

The standard counting of degrees of freedom for a general theory with first and second class constraints is a well known problem. See, e.g., [30] and references therein. For the reasons outlined above, this problem becomes considerably harder for CS theories. In the remaining of this appendix, we examine the Hamiltonian structure of CS systems in order to address this issue.

A.1 Hamiltonian Analysis

From a dynamical point of view, CS systems are described by lagrangians of the form[19]

$$L_{2n+1} = l_a^i(A_j^b)\dot{A}_i^a - A_o^a K_a, \tag{159}$$

where a dot ($\dot{}$) represents a time derivative, and

$$K_a = -\frac{1}{2^n n}\gamma_{aa_1....a_n}\epsilon^{i_1\cdots i_{2n}}F_{i_1 i_2}^{a_1}\cdots F_{i_{2n-1}i_{2n}}^{a_n}.$$

Splitting spacetime into a $2n$-dimensional space plus time, the field equations read

$$\Omega_{ab}^{ij}(\dot{A}_j^b - D_j A_0^b) = 0, \tag{160}$$

$$K_a = 0, \tag{161}$$

[19]Note that in this section, for notational simplicity, we assume the spacetime to be $(2n+1)$-dimensional.

where

$$\Omega^{ij}_{ab} = \frac{\delta l^j_b}{\delta A^a_i} - \frac{\delta l^i_a}{\delta A^b_j} \qquad (162)$$

$$= -\frac{1}{2^{n-1}}\gamma_{aba_2....a_n}\epsilon^{iji_3...i_{2n}}F^{a_2}_{i_3i_4}\cdots F^{a_n}_{i_{2n-1}i_{2n}}$$

is the **symplectic form**. The passage to the Hamiltonian has the problem that the velocities appear linearly in the lagrangian and therefore there are a number of primary constraints

$$\phi^i_a \equiv p^i_a - l^i_a \approx 0. \qquad (163)$$

Besides, there are secondary constraints $K_a \approx 0$, which can be combined with the ϕ s into the expressions

$$G_a \equiv -K_a + D_i\phi^i_a. \qquad (164)$$

The complete set of constraints forms a closed Poison bracket algebra,

$$\begin{array}{ll} \{\phi^i_a, \phi^j_b\} & = \Omega^{ij}_{ab} \\ \{\phi^i_a, G_b\} & = f^c_{ab}\phi^i_c \\ \{G_a, G_b\} & = f^c_{ab}G_c \end{array} ,$$

where f^c_{ab} are the structure constants of the gauge algebra of the theory. Clearly the Gs form a first class algebra which reflects the gauge invariance of the theory, while some of the ϕs are second class and some are first class, depending on the rank of the symplectic form Ω.

A.2 Degeneracy

An intriguing aspect of Chern-Simons theories, not present in most classical systems is the fact that the symplectic form is not a constant matrix but a function of the fields. Hence, the rank of the symplectic form need not be constant. It can change from one region of phase space to another, where the determinant of the matrix Ω^{ij}_{ab} develops zeros of different orders and becomes nonivertible. Regions in phase space with different degrees of degeneracy have different degrees of freedom. If the system reaches a degenerate configuration, some degrees of freedom become frozen in an irreversible process which erases all traces of the initial conditions of the lost degrees of freedom.

This can also be seen from the field equations, which for $D = 2n + 1$, are polynomials of degree n. If some components of the curvature vanish at some point, the remaining curvature factors can take arbitrary values. This corresponds to having some degrees of freedom reduced to mere gauge directions in phase space, and therefore these configurations possess fewer dynamical degrees of freedom.

It can be shown in the context of some simplified mechanical models that the degeneracy of a system generically occur on lower-dimensional submanifolds of phase space. These regions define sets of unstable initial states or sets of stable end points for the evolution [81]. As it was shown in this reference, if the system evolves along an orbit that reaches a surface of degeneracy, Σ, it becomes trapped by the surface and loses the degrees o freedom that correspond to displacements away from Σ. This is an irreversible process which has been observed in the dynamics of vortices described by the Burgers' equation. This equation has a symplectic form which degenerates when two vortices coalesce, an irreversible process that is experimentally observed.

Thus, the space of classical solutions has a very rich structure, describing very distinct dynamical systems in different regions of phase space, all governed by the same action principle. As shown in [76], this phenomenon may be a way to produce dynamical dimensional reduction.

A.3 Counting of Degrees of Freedom

There is a second problem and that is how to separate the first and second class constraints among the ϕs. In Ref.[66] the following results are shown:

- The maximal rank of Ω^{ij}_{ab} is $2n(N-1)$, where N is the number of generators in the gauge Lie algebra.
- There are $2n$ first class constraints among the ϕs which correspond to the generators of spatial diffeomorphisms ($\mathcal{H}_i$).
- The generator of timelike reparametrizations $\mathcal{H}_\perp$ is not an independent first class constraint.

Putting all these facts together one concludes that, in a generic configuration (non degenerate symplectic form), the number of degrees of freedom of the theory is

$$\begin{aligned}
\Delta^{CS} &= (\text{number of coordinates}) - (\text{number of 1st class constraints}) \\
&\quad -\frac{1}{2}(\text{number of 2nd class constraints}) \\
&= 2nN - (N+2n) - \frac{1}{2}(2nN - 2n) \qquad (165) \\
&= nN - N - n. \qquad (166)
\end{aligned}$$

This result is somewhat perplexing. A standard metric (Lovelock) theory of gravity in $D = 2n+1$ dimensions, has

$$\begin{aligned}
\Delta^{Lovelock} &= D(D-3)/2 \\
&= (2n+1)(n-1)
\end{aligned}$$

propagating degrees of freedom [33]. For $D = 2n + 1$, a CS gravity system for the AdS group has $N = D(D + 1)/2 = (2n + 1)(n + 1)$, and therefore,

$$\Delta^{CS} = 2n^3 + n^2 - 3n - 1. \tag{167}$$

In particular, for $D = 5$, $\Delta^{CS} = 13$, while $\Delta^{Lovelock} = 5$. The extra degrees of freedom correspond to propagating modes in ω_μ^{ab}, which in the CS theory are independent from the metric ones, contained in e_μ^a.

In [66] it was also shown that an important simplification occurs when the group has an invariant abelian factor. In that case, the symplectic matrix Ω_{ab}^{ij} takes a partially block-diagonal form where the kernel has the maximal size allowed by a generic configuration. It is a nice surprise in the cases of CS supergravities discussed above that for certain unique choices of $\mathcal{N}$, the algebras develop an abelian subalgebra and make the separation of first and second class constraints possible (e.g., $\mathcal{N}$= 4 for $D = 5$, and $\mathcal{N}$= 32 for $d = 11$). In some cases the algebra is not a direct sum but an algebra with an abelian central extension ($D = 5$). In other cases, the algebra is a direct sum, but the abelian subgroup is not put in by hand but it is a subset of the generators that decouple from the rest of the algebra ($D = 11$).

A.4 Irregularity

Applying the counting of [66] to five-dimensional CS supergravity gave the paradoxical result that the linearized theory around a generic (nondegenerate) background seems to have more degrees of freedom than those in the fully nonlinear regime [72]. This is due to the fact that the first class constraints fail to be functionally independent on some regions of the configuration space. This second type of degeneracy that makes the linearized approximation inadequate is called **irregularity**, and has been discussed in [88]. In this reference, it is shown that there exist two types of irregularities:

Type I, those occurring when a constraint is the product of two or more regular, independent constraints (typically of the form $\Phi = \varphi_1\varphi_2 \approx 0$, with φ_1, φ_1 regular constraints) .

Type II, those in which the gradient of a constraint vanishes on the constraint surface (typically, $\Phi = \varphi^k \approx 0$, where φ is a regular constraint).

Type I are regularizable in the sense that they can be replaced by dynamically equivalent regular systems on different patches of phase space. In these systems the evolution is regular and smooth across boundaries from one patch to the other. Type II irregularities, instead, are simply ill-defined systems which cannot be consistently regularized.

Chern-Simons systems are of type I. The paradox encountered in [72] originates in an unfortunate choice of background corresponding to a configuration where two or more constraints overlap failing to be independent. The linearized analysis gives the wrong because some constraints disappear from the system, which looks as if there were fewer constraints than those actually present.

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