Recursive construction of $D+2$ spacetime dimensional Clifford algebras from the D-dimensional ones.

Let γ_i's denotes the d-dimensional Gamma matrices of a $D = p+q$ spacetime with (p,q) signature. $2d$-dimensional $D+2$ Gamma matrices (denoted as Γ_j) of a $D+2$ spacetime are produced according to either

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_d \\ -1_d & 0 \end{pmatrix}, \quad \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix}$$

$$(p,q) \mapsto (p+1, q+1). \tag{1}$$

or

$$\Gamma_j \equiv \begin{pmatrix} 0 & \gamma_i \\ -\gamma_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix}, \quad \begin{pmatrix} 1_d & 0 \\ 0 & -1_d \end{pmatrix}$$

$$(p,q) \mapsto (q+2, p). \tag{2}$$

Remark 1. The two-dimensional real-valued Pauli matrices τ_A, τ_1, τ_2 which realize the Clifford algebra $C(2,1)$ are obtained by applying either (??) or (??) to the number 1, i.e. the one-dimensional realization of $C(1,0)$. We have indeed

$$\tau_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Remark 2. All Clifford algebras are obtained by recursively applying the algorithms (??) and (??) to the Clifford algebra $C(1,0)$ ($\equiv 1$) and the Clifford algebras of the series $C(0, 3+4m)$ (m non-negative integer), which must be previously known. This is in accordance with the scheme illustrated in the table below.

Table with the maximal Cli®ord algebras (up to d = 256).

1	¤	2	¤	4	¤	8	¤	16	¤	32	¤	64	¤	128	¤	256	¤
(1; 0)	)	(2; 1)	)	(3,2)	)	(4,3)	)	(5,4)	)	(6,5)	)	(7,6)	)	(8,7)	)	(9,8)	)
						(1,4)	!	(2,5)	!	(3,6)	!	(4,7)	!	(5,8)	!	(6,9)	!
					%												
				(0,3)													
					&												
						(5,0)	!	(6,1)	!	(7,2)	!	(8,3)	!	(9,4)	!	(10,5)	!
								(1,8)	!	(2,9)	!	(3,10)	!	(4,11)	!	(5,12)	!
							%										
						(0,7)											
							&										
								(9,0)	!	(10,1)	!	(11,2)	!	(12,3)	!	(13,4)	!
														(1,12)	!	(2,13)	!
													%				
												(0,11)					
													&				
														(13,0)	!	(14,1)	!
																(1,16)	!
															%		
														(0,15)			
															&		
																(17,0)	!

Remark 1: The columns are labeled by the matrix size d of the maximal Cli®ord algebras. Their signature is denoted by the (p; q) pairs.

Remark 2: The underlined Cli®ord algebras in the table are called the primitive maximal Cli®ord algebras. The remaining maximal Cli®ord algebras appearing in the table are the maximal descendent Cli®ord algebras. They are obtained from the primitive maximal Cli®ord algebras by iteratively applying the two recursive algorithms i) and ii).

Remark 3: Any non-maximal Cli®ord algebra is obtained from a given maximal Clifford algebra by deleting a certain number of Gamma matrices.

Remark 4: Cli®ord algebras in even-dimensional spacetimes are always non-maximal.

Example: Explicit construction of the $D = p+q$ spacetime dimensional Clifford algebras for $D = 11$.

(p,q)	type	d
(11,0)	$\subset$ (11,2)	64
(10,1)	M	32
(9,2)	$\subset$ (11,2)	64
(8,3)	M	64
(7,4)	$\subset$ (7,6)	64
(6,5)	M	32
(5,6)	$\subset$ (7,6)	64
(4,7)	M	64
(3,8)	$\subset$ (3,10)	64
(2,9)	M	32
(1,10)	$\subset$ (3,10)	64
(0,11)	M	32

Comments: the maximal Clifford algebras are labeled by M.
The size of the matrix representation is given by the number on the right (d).

Explicit construction of the primitive maximal Clifford algebras of the quaternionic series $C(0,3+8n)$ and the octonionic series $C(0,7+8n)$.

With the help of the three Pauli matrices (??) we construct at first the 4×4 matrices realizing the Clifford algebra $C(0,3)$ and the 8×8 matrices realizing the Clifford algebra $C(0,7)$. They are given, respectively, by

$$C(0,3) \equiv \begin{array}{c} \tau_A\otimes\tau_1, \\ \tau_A\otimes\tau_2, \\ 1_2\otimes\tau_A. \end{array} \qquad (4)$$

and

$$C(0,7) \equiv \begin{array}{c} \tau_A\otimes\tau_1\otimes 1_2, \\ \tau_A\otimes\tau_2\otimes 1_2, \\ 1_2\otimes\tau_A\otimes\tau_1, \\ 1_2\otimes\tau_A\otimes\tau_2, \\ \tau_1\otimes 1_2\otimes\tau_A, \\ \tau_2\otimes 1_2\otimes\tau_A, \\ \tau_A\otimes\tau_A\otimes\tau_A. \end{array} \qquad (5)$$

The three matrices of $C(0,3)$ will be denoted as $\overline{\tau}_i$, $= 1,2,3$. The seven matrices of $C(0,7)$ will be denoted as $\tilde{\tau}_i$, $i = 1,2,\ldots,7$.

Comment. By applying the (??) algorithm to $C(0,7)$ we construct the 16×16 matrices realizing $C(1,8)$ (the matrix with positive signature is denoted as γ_9, ${\gamma_9}^2 = 1$, while the eight matrices with negative signatures are denoted as γ_j, $j = 1,2\ldots,8$, with ${\gamma_j}^2 = -1$).

We are now in the position to explicitly construct the whole series of primitive maximal Clifford algebras $C(0,3+8n)$, $C(0,7+8n)$ through the formulas

$$C(0,3+8n) \equiv \begin{array}{ll} \tau_i \otimes \gamma_9 \otimes \ldots & \ldots \quad \ldots \otimes \gamma_9, \\ 1_4 \otimes \gamma_j \otimes 1_{16} \otimes \ldots & \ldots \quad \ldots \otimes 1_{16}, \\ 1_4 \otimes \gamma_9 \otimes \gamma_j \otimes 1_{16} \otimes \ldots & \ldots \quad \ldots \otimes 1_{16}, \\ 1_4 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes 1_{16} \otimes \ldots & \ldots \quad \ldots \otimes 1_{16}, \\ \ldots & \ldots \quad \ldots, \\ 1_4 \otimes \gamma_9 \otimes \ldots & \ldots \quad \otimes \gamma_9 \otimes \gamma_j, \end{array} \qquad (6)$$

and similarly

$$C(0,7+8n) \equiv \begin{array}{ll} \tau_i \otimes \gamma_9 \otimes \ldots & \ldots \quad \ldots \otimes \gamma_9, \\ 1_8 \otimes \gamma_j \otimes 1_{16} \otimes \ldots & \ldots \quad \ldots \otimes 1_{16}, \\ 1_8 \otimes \gamma_9 \otimes \gamma_j \otimes 1_{16} \otimes \ldots & \ldots \quad \ldots \otimes 1_{16}, \\ 1_8 \otimes \gamma_9 \otimes \gamma_9 \otimes \gamma_j \otimes 1_{16} \otimes \ldots & \ldots \quad \ldots \otimes 1_{16}, \\ \ldots & \ldots \quad \ldots, \\ 1_8 \otimes \gamma_9 \otimes \ldots & \ldots \quad \otimes \gamma_9 \otimes \gamma_j, \end{array} \qquad (7)$$

Comment. The tensor product of the 16-dimensional representation is taken n times. The total size of the (??) matrix representations is then 4×16^n, while the total size of (??) is 8×16^n.